121. Some Trigonometrical Series. XVI

By Shin-ichi IZUMI
Mathematical Institute, Tokyo Metropolitan University, Tokyo
(Comm. by Z. SUETUNA, M.J.A., Oct. 12, 1955)

N. Wiener [1] proposed the problem to find the condition of the convergence of the series

$$\sum_{n=1}^{\infty} |s_n(x)-f(x)|, \tag{1}$$

and

$$\sum_{n=1}^{\infty} (s_n(x)-f(x))^2, \tag{2}$$

where $s_n(x)$ is the nth partial sum of Fourier series of $f(x)$. The uniform convergence of (2) was treated in [2].

The object of this paper is to find the condition of almost everywhere convergence of the series

$$\sum_{n=1}^{\infty} |s_n(x)-f(x)|^\lambda. \tag{3}$$

In the case $\lambda=1$, that is, in (1) T. Tsuchikura [3] has gotten the condition by the Fourier coefficients of $f(x)$. We prove the following

Theorem. *Let $p>1$, $p\geqq\lambda\geqq1$, and ε be any positive number. If $f(x)$ is of the power series type*[1] *and*

$$\left(\int_0^{2\pi} |f(x+t)-f(x)|^p\,dx\right)^{1/p} \leqq At^{1/\lambda}\Big/\left(\log\frac{1}{t}\right)^{(1+\varepsilon)/\lambda} \tag{4}$$

then the series (3) converges almost everywhere.

In the proof we use the technic due to A. Zygmund [4] and his lemma:

Lemma. *Suppose that $p>1$ and*

$$\Big\|\sum_{\nu=m}^{n}\gamma_\nu e^{i\nu x}\Big\|_p \leqq C$$

where $\|\ \|_p$ denotes the L^p-norm and suppose that

$$|\lambda_\nu|\leqq M,\ \sum_{\nu=m}^{n-1}|\lambda_\nu-\lambda_{\nu+1}|\leqq M,$$

then

$$\Big\|\sum_{\nu=m}^{n}\gamma_\nu\lambda_\nu e^{i\nu x}\Big\|_p \leqq A_p MC.$$

Let us now prove the theorem. It is sufficient to prove that the series

$$\sum_{n=1}^{\infty}\int_0^{2\pi} |s_n(x)-f(x)|^\lambda\,dx \tag{5}$$

is convergent. Then

1) This condition is implied by (4) when $\lambda>1$.

$$(6)\qquad \int_0^{2\pi} |s_k(x)-f(x)|^\lambda\, dx \leqq \|s_k(x)-f(x)\|_p^\lambda.$$

Now let

$$f(x)\sim \sum_{\nu=1}^{\infty} c_\nu e^{i\nu x},$$

then, by (4) with t replaced by $2t$,

$$\Big\| \sum_{\nu=1}^{\infty} c_\nu e^{i\nu x}\sin \nu t \Big\|_p \leqq At^{1/\lambda}\Big/\Big(\log \frac{1}{t}\Big)^{(1+\varepsilon)/\lambda}.$$

By the Riesz theorem

$$\Big\| \sum_{\nu=2^h}^{2^{h+1}-1} c_\nu e^{i\nu x}\sin \nu t \Big\|_p \leqq At^{1/\lambda}\Big/\Big(\log \frac{1}{t}\Big)^{(1+\varepsilon)/\lambda}.$$

If we take $t=\pi/2^{h+2}$, then by Lemma we get

$$\Big\| \sum_{\nu=2^h}^{2^{h+1}-1} c_\nu e^{i\nu x} \Big\|_p \leqq A/2^{h/\lambda} h^{(1+\varepsilon)/\lambda}.$$

This estimation holds even if the lower limit of the left side summation is replaced by m such that

$$2^h < m < 2^{h+1}-1,$$

and the upper limit by ∞. Hence (5) is less than

$$A\sum_{h=1}^{\infty} 2^h/(2^{h/\lambda} h^{(1+\varepsilon)/\lambda})^\lambda \leqq A\sum_{h=1}^{\infty} \frac{1}{h^{1+\varepsilon}} < \infty.$$

Thus the theorem is proved.

References

[1] N. Wiener: Tauberian theorems, Ann. Math., **32** (1933).
[2] S. Izumi: Some trigonometrical series. XIII, Proc. Japan Acad., **31** (1955).
[3] T. Tsuchikura: To appear in the Tôhoku Math. Journ.
[4] A. Zygmund: Modulus of continuity of functions, Revista Math. (1952).