164. On Countably Paracompact Spaces

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This note will give a characterization of a countably paracompact space, i.e. a topological space X which has a locally finite refinement for every countable open covering.

Theorem: In order that a topological space X be countably paracompact it is necessary and sufficient that:

A. If a decreasing sequence $\{F_i\}$ of non empty closed sets F_i with vacuous intersection is given, then there exists a decreasing sequence $\{G_i\}$ of open sets such that their closure $\overline{G_i}$ have a vacuous intersection and $G_i \supset F_i$.

Proof. If X is a countably paracompact space and given $\{F_i\}$, then $\{X-F_i\}$ is a countable open covering of X, therefore it has a locally finite refinement $\mathfrak{W}$. For each open set W of $\mathfrak{W}$ let $g(W)$ be the first $X-F_i$ containing W, and let V_i be the union of all W for which $g(W)=X-F_i$. Then V_i is open and $V_i \subset X-F_i$, $\{V_i\}$ is a locally finite covering of X.

Put: $G_i = \sum_{n=i+1}^{\infty} V_n$. Then G_i is open,

$$G_i \supset X-(V_1+\cdots+V_i) \supset X-(X-F_i)=F_i,$$

hence $G_i \supset F_i$.

For every point x of X there exists a neighborhood $u(x)$ such that it meets only a finite number of V_i, since $\{V_i\}$ is a locally finite. Therefore there exists an i such that $u(x) \frown \sum_{n=i+1}^{\infty} V_n=0$, that is, $x \bar{\in} \overline{G_i}$. Accordingly $\pi\overline{G_i}=0$.

Conversely let X be a space with the condition A, and let $\{U_i\}$ be a given countable open covering of X.

Put: $F_i = X - \sum_{n=1}^{i} U_n$. Then by the condition A there exist open sets G_i such that $G_i \supset F_i$,

$$G_1 \supset G_2 \ldots \quad \text{and} \quad \pi\overline{G_i}=0.$$

Put: $X-G_i=E_i$. Then E_i is obviously closed and $E_i \subset \sum_{n=1}^{i} U_n$.

Finally put: $V_i = U_i - E_{i-1}$. Then V_i is clearly open and $V_i \subset U_i$. Moreover, since $V_i = U_i - E_{i-1} \supset U_i - \sum_{n=1}^{i-1} U_n$, we have $\sum_{i=1}^{\infty} V_i \supset \sum_{i=1}^{\infty}(U_i - \sum_{n=1}^{i-1} U_n) = \sum_{i=1}^{\infty} U_i = X$, thus $\{V_i\}$ is a refinement of $\{U_i\}$. To each point x of X, we choose the first i such that $x \bar{\in} \overline{G_i}$. Then there exists a

neighborhood $u(x)$ of x such that $u(x) \frown G_i = 0$, hence $u(x) \subset E_i$. Therefore $u(x) \frown V_{i+j} = 0$　$j = 1, 2, \ldots$, thus $u(x)$ meets a finite number of V_i. Therefore $\{V_i\}$ is a locally finite covering of X. X is thus proved to be countably paracompact.

　　Corollary: In order that each countable open covering of X have a point finite refinement, it is necessary and sufficient that, if a decreasing sequence $\{F_i\}$ of non empty closed sets with vacuous intersection is given, then there exists a decreasing sequence $\{G_i\}$ of open sets with vacuous intersection and $G_i \supset F_i$. The proof runs exactly as above. Instead of a neighborhood $u(x)$ of x we have only to consider the point x itself.

　　Hence the two spaces are equivalent in normal space.