158. Cohomology of the p-fold Cyclic Products

By Minoru NAKAOKA
Osaka City University, Osaka

(Comm. by K. KUNUGI, M.J.A., Dec. 12, 1955)

In this paper, we study certain cohomological properties of the p-fold cyclic product of a finite complex.[1] Our approach is based on the studies of R. Thom-W. T. Wu[2] who gave an intrinsic definition of the Steenrod reduced powers. We state here only the results without proofs. Full details, together with an easy complete treatment of the Thom-Wu theory, will appear in a short-coming paper.[3]

§1. Let W be a finite simplicial complex, and let t be a transformation on W satisfying the following conditions: i) t is a simplicial map of period p; ii) if a simplex is mapped onto itself by t, it remains pointwise fixed. Let F be the set of fixed points under t, then F is a subcomplex of W. Denote by $C^r(W,F;G)$ the r-cochain group of (W,F) with coefficients in an abelian group G, and define cochain maps $\sigma,\tau: C^r(W,F;G)\longrightarrow C^r(W,F;G)$ by

$$\sigma=\sum_{i=0}^{p-1} t^{i\#},\qquad \tau=1-t^{\#}$$

respectively, where $t^{\#}$ is the cochain map induced by t. We shall also denote these maps by ρ and $\bar\rho$ agreeing that ρ may stand for σ, $\bar\rho$ for τ or *vice versa*. Then the sequence $\{\rho C^r(W,F;G),\delta\}$ of the image groups $\rho C^r(W,F;G)$ and the coboundary homomorphisms δ form a cochain complex. The cohomology group of this complex will be denoted by ${}^{\rho}H^r(W,F;G)$. Then we have the Smith-Richardson exact sequence

$$\cdots\longrightarrow {}^{\bar\rho}H^r(W,F;G)\xrightarrow{\alpha_\rho} H^r(W,F;G)\xrightarrow{\beta_\rho} {}^{\rho}H^r(W,F;G)$$
$$\xrightarrow{\gamma_\rho} {}^{\bar\rho}H^{r+1}(W,F;G)\longrightarrow\cdots .$$

Let W_t be the orbit space over W relative to t, and $\pi: W\longrightarrow W_t$ the natural projection. Then W_t is a simplicial complex having $F_t=\pi(F)$ as a subcomplex, and we have

$$I^*: H^r(W_t,F_t;G)\approx {}^{\sigma}H^r(W,F;G),\qquad \pi^*: H^r(F_t;G)\approx H^r(F;G),$$

where both I^* and π^* are homomorphisms induced by π. Furthermore we have the following results;

1) Throughout this paper, p denotes an arbitrarily fixed prime integer.

2) R. Thom: Une théorie intrinsèque des puissances de Steenrod, Strasbourg Colloq., 1951 (mimeographed); W. T. Wu: Sur les puissances de Steenrod, ibid.

3) M. Nakaoka: Cohomology theory of a complex with a transformation of prime period and its applications, to appear in J. Inst. Polyt., Osaka City Univ.

(1.1) *Let* $\pi^*: H^r(W_t; G) \longrightarrow H^r(W; G)$ *be a homomorphism induced by* π. *If* $a \in H^r(W_t; G)$ *and* $\pi^*(a)=0$, *then* $pa=0$.
(1.2) *Let* G *be a field of characteristic* q, *not a divisor of* p. *Then* π^* *in (1.1) is isomorphic into, and its image is* $\sigma^* H^r(W; G)$.

Write $\mu = I^{*^{-1}} \gamma_\tau \gamma_\sigma I^*$, then we have a homomorphism

$$\mu: H^r(W_t; F_t, G) \longrightarrow H^{r+2}(W_t, F_t; G).$$

Since $p\sigma=\sigma^2$, the inclusion defines a homomorphism $\psi: {}^\sigma H^r(W, F; G) \longrightarrow {}^\tau H^r(W, F; G_p)$, where G_p denotes the factor group G/pG. We may now consider a homomorphism

$$\nu: H^r(W_t, F_t; G) \longrightarrow H^{r+1}(W_t, F_t; G_p)$$

defined by $\nu = I^{*^{-1}} \gamma_\tau \psi I^*$. Define next a homomorphism $\phi: C^r(W; G) \longrightarrow C^r(W_t; G)$ by $\pi\phi=\sigma$. Then ϕ is a cochain map, and hence it induces a homomorphism

$$\phi^*: H^r(W; G) \longrightarrow H^r(W_t; G).$$

Let $\eta: G \longrightarrow G_p$ be the natural projection, then $\eta\phi C^r(W; G) \subset C^r(W_t, F_t; G_p)$. Therefore, ϕ induces a homomorphism

$$\phi_0^*: H^r(W; G) \longrightarrow H^r(W_t, F_t; G_p).$$

Let $j^*: H^r(W_t, F_t; G_p) \longrightarrow H^r(W_t; G_p)$ be the inclusion, then $j^*\phi_0^* = \eta\phi^*$ is obvious.

The homomorphisms μ, ν, and ϕ_0^* are of importance in our study. Let $i^*: H^r(W; G) \longrightarrow H^r(F; G)$ be the inclusion, and $\delta^*: H^r(F_t; G) \longrightarrow H^{r+1}(W_t, F_t; G)$ the coboundary homomorphism, then we have

(1.3) $\nu^2=0$ *if* $p \geqq 3$, *and* $=\eta\mu$ *if* $p=2$; $\mu\nu=\nu\mu$.

(1.4) $\nu\phi_0^*=0$ *if* $p\geqq 3$, *and* $=\eta\delta^*\pi^{*^{-1}}i^*$ *if* $p=2$; $\mu\phi_0^* = -\nu\delta^*\pi^{*^{-1}}i^*$.

From here to (3.3), the coefficients group for cohomology will be the group Z_p of integers mod p. Let $a', b' \in H^*(W; Z_p)$, $a, b \in H^*(W_t, F_t; Z_p)$, and denote by $\smile$ the cup product. Then we have

(1.5) $\mu^\alpha(a) \smile \mu^\beta(b) = \mu^{\alpha+\beta}(a \smile b)$, $(\alpha, \beta \geqq 0)$,
$\mu^\alpha(a) \smile \nu(b) = (-1)^{\dim a} \mu^\alpha \nu(a \smile b)$, $(\alpha \geqq 0)$,
$\nu(a) \smile \nu(b) = 0$ *if* $p \geqq 3$, *and* $=\mu(a \smile b)$ *if* $p=2$.

(1.6) $\phi_0^*(a' \smile \sigma^* b') = \phi_0^*(a') \smile \phi_0^*(b')$,
$\nu(\phi_0^* a' \smile \phi_0^* b')=0$, $\mu(\phi_0^* a' \smile \phi_0^* b')=0$.

Let (X, A) be any pair of a simplicial complex X and its subcomplex A, and let

$$\mathcal{P}^s: H^q(X, A; Z_p) \longrightarrow H^{q+2s(p-1)}(X, A; Z_p), \ (p \geqq 3),$$
$$Sq^i: H^q(X, A; Z_2) \longrightarrow H^{q+i}(X, A; Z_2),$$
$$\Delta_p: H^q(X, A; Z_p) \longrightarrow H^{q+1}(X, A; Z_p),$$

be the Steenrod reduced power, the Steenrod square and the Bockstein homomorphism respectively.[4] Then we have the following:

(1.7) $\mathcal{P}^s\mu - \mu\mathcal{P}^s = \mu^p \mathcal{P}^{s-1}$, $\mathcal{P}^s \nu = \nu \mathcal{P}^s$, $Sq^i\nu - \nu Sq^i = \nu^2 Sq^{i-1}$.

(1.8) $\phi_0^* \mathcal{P}^s - \mathcal{P}^s \phi_0^* = \mu^{p-1} \mathcal{P}^{s-1} \phi_0^*$, $\phi_0^* Sq^i - Sq^i \phi_0^* = \nu Sq^{i-1} \phi_0^*$.

4) N. E. Steenrod: Cyclic reduced powers of cohomology classes, Proc. Nat. Acad. Sci., U. S. A., **39** (1953).

(1.9) $\Delta_p\nu+\nu\Delta_p=\mu, \quad \mu\Delta_p=\Delta_p\mu.$

(1.10) $\phi_0^*\Delta_p-\Delta_p\phi_0^*=\delta^*\pi^{*-1}i^*.$

We shall prove in the paper[1] the above formulas by making only use of the elementary simplicial cohomology theory.[5]

§ 2. Let K be a finite simplicial complex, then we shall denote by $\boldsymbol{X}(K)$ the p-fold Cartesian product of K. Let $T:\boldsymbol{X}(K)\longrightarrow\boldsymbol{X}(K)$ be the map defined by the cyclic permutation of coordinates. Then T is a transformation on $\boldsymbol{X}(K)$, and satisfies the conditions i) and ii) in § 1 for an appropriate simplicial decomposition of $\boldsymbol{X}(K)$. Therefore we may apply the results in § 1 with $W=\boldsymbol{X}(K)$ and $t=T$. The orbit space over $\boldsymbol{X}(K)$ relative to T is called *the p-fold cyclic product of K*, and will be denoted by $\boldsymbol{Z}(K)$ in the following. The set of fixed points under T is the diagonal $\boldsymbol{D}(K)=\{(x,x,\dots,x)\mid x\in K\}$. Write $\boldsymbol{d}(K)=\pi\boldsymbol{D}(K)$, where $\pi:\boldsymbol{X}(K)\longrightarrow\boldsymbol{Z}(K)$ is the projection, and define a homeomorphism $d_0:K\longrightarrow\boldsymbol{d}(K)$ by $d_0(x)=\pi(x,x,\dots,x)(x\in K)$. Denote by $N^r(\boldsymbol{Z}(K),\boldsymbol{d}(K);Z_p)$ the kernel of $\pi^*:H^r(\boldsymbol{Z}(K),\boldsymbol{d}(K);Z_p)\longrightarrow H^r(\boldsymbol{X}(K),\boldsymbol{D}(K);Z_p)$, and define a homomorphism $E_s:H^q(K;Z_p)\longrightarrow H^{q+s}(\boldsymbol{Z}(K),\boldsymbol{d}(K);Z_p)$ $(s>0)$ by

$$E_{2\alpha+1}=\mu^\alpha\delta^*d_0^{*-1},\quad E_{2\alpha+2}=\mu^\alpha\nu\delta^*d_0^{*-1}.$$

Then we have[2]

(2.1) *E_s is isomorphic into for $1\leqq s\leqq(p-1)q$.*

(2.2) *$N^r(\boldsymbol{Z}(K),\boldsymbol{d}(K);Z_p)=\sum_{s=m}^{r-1}E_{r-s}H^s(K;Z_p)$,[6] where m is the greatest integer $\leqq(r+p-1)/p$.*

Let $\Omega(K;Z_p)$ be a homogeneous base for the vector space $H^*(K;Z_p)$, then the cross product $b_1\times b_2\times\cdots\times b_p(b_i\in\Omega(K;Z_p))$ is an element of $H^*(\boldsymbol{X}(K);Z_p)$. Consider a set $\mathfrak{B}_r(\Omega(K;Z_p))=\{b_1\times b_2\times\cdots\times b_p\mid b_i\in\Omega(K;Z_p),\sum_{i=1}^p\dim b_i=r\}$ and its subset $\mathfrak{B}_r''(\Omega(K);Z_p)=\{b\times b\times\cdots\times b\mid b\in\Omega(K;Z_p),\, p\dim b=r\}$, and let $\mathfrak{B}_r'(\Omega(K;Z_p))=\mathfrak{B}_r(\Omega(K;Z_p))-\mathfrak{B}_r''(\Omega(K;Z_p))$. Then it is well known that $\mathfrak{B}_r(\Omega(K;Z_p))$ is a base for $H^r(\boldsymbol{X}(K);Z_p)$. Let $\mathfrak{B}'^r(\Omega(K;Z_p))\subset H^r(\boldsymbol{X}(K);Z_p)$ be a vector subspace spanned by $\mathfrak{B}_r'(\Omega(K;Z_p))$. Then we have

(2.3) $H^r(\boldsymbol{Z}(K),\boldsymbol{d}(K);Z_p)=N^r(\boldsymbol{Z}(K),\boldsymbol{d}(K);Z_p)+\phi_0^*(\mathfrak{B}'^r(\Omega(K;Z_p)))$,[6] *and the kernel of ϕ_0^* is $\tau^*\mathfrak{B}'^r(\Omega(K;Z_p))$.*

It follows from (2.2) and (2.3) that any element of $H^r(\boldsymbol{Z}(K),\boldsymbol{d}(K);Z_p)$ can be represented as a linear combination of elements with types $E_s(x)$ and $\phi_0^*(x_1\times x_2\times\cdots\times x_p)$, where $x,x_i\in H^*(K;Z_p)$.

We shall next consider the operations $\mathcal{P}^s$, Δ_p, and $\smile$ in $H^*(\boldsymbol{Z}(K),\boldsymbol{d}(K);Z_p)$. The formulas (1.3)–(1.10) yield the following:

---

5) Some of the formulas (1.3)–(1.10) are proved by Thom by making use of a multiplicative property in the Cartan-Leray cohomology theory (see Reference 2). As for the formulas for $p=2$, see also R. Bott: On symmetric products and the Steenrod squares, Ann. Math., **57** (1953).

6) This sum is direct.

(2.4) i) $\Delta_p\phi_0^*(x_1\times x_2\times\cdots\times x_p)$
$=\phi_0^*(\Delta_p(x_1\times x_2\times\cdots\times x_p))-E_1(x_1\smile x_2\smile\cdots\smile x_p)$.

ii) $\mathcal{P}^s\phi_0^*(x_1\times x_2\times\cdots\times x_p)=\phi_0^*(\mathcal{P}^s(x_1\times x_2\times\cdots\times x_p))$
$+\sum_{j=1}^{s}(-1)^{j+1}E_{2j(p-1)}\mathcal{P}^{s-j}(x_1\smile x_2\smile\cdots\smile x_p)$.

iii) $Sq^t\phi_0^*(x_1\times x_2)=\phi_0^*(Sq^t(x_1\times x_2))+\sum_{j=1}^{t}E_jSq^{t-j}(x_1\smile x_2)$.

(2.5) i) $\Delta_pE_{2a+1}(x)=-E_{2a+1}(\Delta_px)$, $\Delta_pE_{2a+2}(x)=E_{2a+3}(x)+E_{2a+2}(\Delta_px)$.

ii) $\mathcal{P}^sE_{2a+1}(x)=\sum_{j=0}^{s}{}_aC_{s-j}\,E_{2(s-j)(p-1)+2a+1}(\mathcal{P}^jx)$,
$\mathcal{P}^sE_{2a+2}(x)=\sum_{j=0}^{s}{}_aC_{s-j}\,E_{2(s-j)(p-1)+2a+2}(\mathcal{P}^jx)$.

iii) $Sq^tE_{a+1}(x)=\sum_{j=0}^{t}{}_aC_{t-j}\,E_{a+1+t-j}(Sq^jx)$.[7]

(2.6) i) $\phi_0^*(x_1\times x_2\times\cdots\times x_p)\smile\phi_0^*(y_1\times y_2\times\cdots\times y_p)$
$=\sum_{j=1}^{p}(-1)^{\varepsilon_j}\phi_0^*((x_1\smile y_j)\times(x_2\smile y_{j+1})\times\cdots\times(x_p\smile y_{j-1}))$,

where $\varepsilon_j=(1+\sum_{i=1}^{p}\dim y_i)(\sum_{i=1}^{j}\dim y_i)+\sum_{a=0}^{p-2}(\dim y_{a+j}(\sum_{\beta=a+2}^{p}\dim x_\beta^{\#}))$.

ii) $E_s(x)\smile\phi_0^*(x_1\times x_2\times\cdots\times x_p)=0$, $E_s(x)\smile E_t(y)=0$.

(2.7) i) *Let* $p\geqq 3$ *and* $\dim x=q$, *then*
$\phi_0^*(x\times x\times\cdots\times x)=\chi_q\sum_{0\leq j<q/2}(-1)^jE_{(p-1)(q-2j)}\mathcal{P}^j(x)$,
$E_{(p-1)q+1}(x)=\sum_{0<j\leq q/2}(-1)^{j+1}E_{(p-1)(q-2j)+1}\mathcal{P}^j(x)$
$+\sum_{0\leq j<q/2}(-1)^{j+1}E_{(p-1)(q-2j)}\Delta_p\mathcal{P}^j(x)$,

where χ_q *is a certain non-zero integer* mod p, *depending only on* q.[8]

ii) $\phi_0^*(x\times x)=\sum_{j=0}^{q-1}E_{q-j}Sq^j(x)$, $E_{q+1}(x)=\sum_{j=0}^{q}E_{q-j+1}Sq^j(x)$.

Since we have for $r>1$ an exact sequence

$$0\longrightarrow H^{r-1}(\boldsymbol{d}(K);Z_p)\xrightarrow{\delta^*}H^r(\boldsymbol{Z}(K),\boldsymbol{d}(K);Z_p)\xrightarrow{j^*}H^r(\boldsymbol{Z}(K);Z_p)\longrightarrow 0,$$

the cohomology of $\boldsymbol{Z}(K)$[9] can be determined immediately from that of $(\boldsymbol{Z}(K),\boldsymbol{d}(K))$ above-mentioned. For example, as for the rank of the group $H^r(\boldsymbol{Z}(K);Z_p)$ we have

(2.8) *Denote by* $R_r(Y;p)$ *the rank of* $H^r(Y;Z_p)$, *then*

$$R_r(\boldsymbol{Z}(K);p)=\sum_{m\leq s\leq r-2}R_s(K;p)+1/p\{R_r(\boldsymbol{X}(K),p)-R_{r/p}(K,p)\},$$

where m *is the same as in (2.2), and it is to be read that* $R_{r/p}(K,p)=0$ *if* r *is not divisible by* p.

§3. As a special case, we shall consider the cohomology of the p-fold cyclic product of a sphere.[10] Let S^n be an n-sphere, and let $e_n^{\#}\in H^n(S^n;Z_p)$ be a generator. Write $a_s^{\#}=j^*E_{s-n}(e_n^{\#})\in H^s(\boldsymbol{Z}(S^n);Z_p)$ for $n+2\leqq s\leqq np$. Given a set $\{\alpha_1,\alpha_2,\ldots,\alpha_q\}$ of $q(1\leqq q\leqq p)$ different integers mod p, we shall also write

$$g_{nq}^{\#}(\alpha_1,\alpha_2,\ldots,\alpha_q)=\phi^*(x_1\times x_2\times\cdots\times x_p)\in H^{nq}(\boldsymbol{Z}(S^n);Z_p),$$

where $x_j=e_n^{\#}$ if $j\equiv\alpha_1,\alpha_2,\ldots,\alpha_q$ mod p, and $=1^{\#}$ otherwise.[11] Then we have the following:

---

7) ${}_aC_\beta$ denotes the binomial coefficient with the usual conventions.

8) If q is even, then $\chi_q=(-1)^{q/2}$.

9) The homology groups of $\boldsymbol{Z}(K)$ for $p=2$ are also calculated in a paper of S. K. Stein: Homology of the two-fold symmetric product, Ann. Math., **59** (1954).

10) Using different methods from ours, this special case is studied in S. D. Liao: On the topology of cyclic products of spheres, Trans. Amer. Math. Soc., **77** (1954).

11) $1^{\#}$ denotes the cohomology class containing the fundamental zero-cocycle.

(3.1) *$a_s^{\#}$ and $g_{nq}^{\#}(\alpha_1, \alpha_2, \dots, \alpha_q)$ are non-zero elements; $g_{nq}^{\#}(\alpha_1, \alpha_2, \dots, \alpha_q) = \pm g_{nq}^{\#}(\beta_1, \beta_2, \dots, \beta_q)$ if and only if there is an integer k such that $\{\alpha_1+k, \alpha_2+k, \dots, \alpha_q+k\} = \{\beta_1, \beta_2, \dots, \beta_q\}$; there is ${}_pC_q/p$ different $g_{nq}^{\#}(\alpha_1, \alpha_2, \dots, \alpha_q)$ for a given q; as a base for $H^*(\boldsymbol{Z}(S^n); Z_p)$, we can take $1^{\#}$, $a_s^{\#}(n+2 \leqq s \leqq np)$, and $g_{nq}^{\#}(\alpha_1, \alpha_2, \dots, \alpha_q)$ for $1 \leqq q \leqq p-1$ and every set $\{\alpha_1, \alpha_2, \dots, \alpha_q\}$; $g_{np}^{\#}(1, 2, \dots, p) = \chi_n a_{np}^{\#}$.*

(3.2) i) $\Delta_p g_{nq}^{\#}(\alpha_1, \alpha_2, \dots, \alpha_q) = 0$; $\Delta_p a_{n+2\alpha+1}^{\#} = 0$, $\Delta_p a_{n+2\alpha+2}^{\#} = a_{n+2\alpha+3}^{\#}$.

ii) $\mathcal{P}^s g_n^{\#}(1) = (-1)^{s+1} a_{n+2s(p-1)}^{\#}$ $(s \neq 0)$,

$\mathcal{P}^s g_{nq}^{\#}(\alpha_1, \alpha_2, \dots, \alpha_q) = 0$ *if* $q > 1$ *and* $s \neq 0$,

$\mathcal{P}^s a_{n+2\alpha+1}^{\#} = {}_\alpha C_s a_{n+2s(p-1)+2\alpha+1}^{\#}$, $\mathcal{P}^s a_{n+2\alpha+2}^{\#} = {}_\alpha C_s a_{n+2s(p-1)+2\alpha+2}^{\#}$.

iii) $Sq^i g_n^{\#}(1) = a_{n+i}^{\#}$ $(i \geqq 2)$, $Sq^i a_{n+\alpha+1}^{\#} = {}_\alpha C_i a_{n+\alpha+i+1}^{\#}$.

We can also determine by (2.6) the cup product in $H^*(\boldsymbol{Z}(S^n); Z_p)$. For example we have

(3.3) *Let $p \geqq 3$, then $g_n^{\#}(1) \smile g_n^{\#}(1) = 2(\sum_{k=2}^{(p+1)/2} g_{2n}^{\#}(1, k))$ for even n, and $=0$ for odd n. Let $p=2$, then $g_n^{\#}(1) \smile g_n^{\#}(1) = a_{2n}^{\#}$.*

Finally we shall determine the integral cohomology group $H^r(\boldsymbol{Z}(S^n); Z)$. Given an abelian group A and a prime number q, we shall denote by $C(A, q)$ the q-primary component of A, and by $C(A, \infty)$ the free component of A. Moreover write $J(A; r)$ for the direct sum of r groups each of which is isomorphic with A. Then, from (1.1), (1.2), and (2.8), we have by the universal coefficient theorem

(3.4) i) $C(H^i(\boldsymbol{Z}(S^n); Z), q) = 0$ *for any i and $q \neq p, \infty$.*

ii) $C(H^i(\boldsymbol{Z}(S^n); Z), \infty) \approx Z$ *for $i=0$ and pn with $(p-1)n = $ even,* $\approx J(Z, {}_pC_q/p)$ *for $i = nq$ with $1 \leqq q \leqq p-1$, $=0$ for other i.*

iii) $C(H^i(\boldsymbol{Z}(S^n), Z); p) \approx Z_p$ *if $i-n$ is odd and $3 \leqq i-n \leqq (p-1)n$, $=0$ for other i.*

Note that the homomorphism $E_{2\alpha+1}$ can be also defined for the integral cohomology groups by the same formula as in § 2. Using this homomorphism $E_{2\alpha+1} = \mu^\alpha \delta^* d_0^{*-1} : H^q(K; Z) \longrightarrow H^{q+2\alpha+1}(\boldsymbol{Z}(K), \boldsymbol{d}(K); Z)$ and the homomorphism $\phi^* : H^r(\boldsymbol{X}(K); Z) \longrightarrow H^r(\boldsymbol{Z}(K); Z)$, we have

(3.5) *Let $e_n^* \in H^n(S^n; Z)$ be a generator. Then $j^* E_{2\alpha+1}(e_n^*)$, is a generator of $C(H^{n+2\alpha+1}(\boldsymbol{Z}(S^n); Z), p)$ for $1 \leqq \alpha \leqq \frac{1}{2}(pn-n-1)$ and $C(H^{nq}(\boldsymbol{Z}(S^n); Z), \infty) = \phi^* H^{nq}(\boldsymbol{X}(S^n); Z)$.*