155. On a Theorem of N. Jacobson

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Recently in his paper [2] N. Jacobson proved the following: If R' is a division ring of characteristic $\neq 2$ which is finite over its center Z' and a division ring R contains R' and has left dimensionality $[R:R']_l=2$ then R is Galois over R'.

In this note we shall extend this result to simple rings.

If R is a *simple ring* (i.e. a primitive ring with minimum condition) then the length of the composition series of the R-module R is denoted by $[R]$. In general, for a finitely generated unitary left R-module M, the length of the composition series of the R-module M is denoted by $[M|R]_l$. As is well known, M possesses a linearly independent R-basis if and only if $[R]$ divides $[M|R]_l$ and the dimensionality $[M:R]_l=[M|R]_l/[R]$.

In the below, that R' is a simple subring of a simple ring R will mean that the subring R' is simple and the identity element of R' is the same with that of R. And R will be said to be *Galois* over R' if 1) R' is an invariant subring of some automorphism group $\mathfrak{G}$ of R, 2) $[\mathfrak{G}:\mathfrak{J}]<\infty$, where $\mathfrak{J}$ is the totality of inner automorphisms in $\mathfrak{G}$, 3) $V(R')$, the centralizer of R' in R, is simple and finite over Z (cf. [4]).

We set first the following

Lemma. *Let R be a simple ring, R' a simple subring of R, Z, and Z' the centers of R and R' respectively. If $[R|R']_l<\infty$ and $[R':Z']<\infty$ then $[R:Z]<\infty$. Conversely, if $[R:Z]<\infty$ then $[R':Z']\leqq[R:Z]$.*

Proof. Let $[R':Z']=g^2$, $[R|R']_l=d$. Then $[R:Z']=gd$. If $\mathfrak{L}$ denotes the Z'-linear transformation ring of the left Z'-module R then $\mathfrak{L}$ contains R_r, all right multiplications by elements of R, and $\mathfrak{L}$ is isomorphic to $(Z')_{gd}$, the ring of $gd\times gd$ matrices over the commutative field Z'. Since, in $(Z')_{gd}$, the polynomial identity $[x_1,\dots,x_{2gd}]=\sum\pm x_{i_1}\dots x_{i_{2gd}}=0$ holds, where the summation runs over all permutations of $(1,\dots,2gd)$ and the sign $+$ and $-$ according as the permutation is even or odd (see [1]), $[x_1,\dots,x_{2gd}]=0$ in R_r, whence also in R. As R is simple, by Theorem 1 of [3], R is of finite rank over Z. By making use of the same method as in the proof of Theorem 1 of [2], we shall obtain the last part.

Now we can extend Jacobson's theorem to simple rings as follows:

Theorem. *Let R be a simple ring, R' a simple subring of R, Z, and Z' the centers of R and R' respectively. If $[R:R']_l=2$, $[R':Z'] < \infty$ and the characteristic of $Z' \neq 2$, then R is Galois over R'.*

Proof. Our lemma shows that $[R:Z]<\infty$. We distinguish two cases: I. $R' \supseteqq Z$. It is well known that $V(V(R'))=R'$ and $V(R')$ is simple. Since each element of a simple ring is represented as a sum of regular elements in the ring, R' is the invariant subring of the inner automorphisms determined by all regular elements of $V(R')$. Clearly R is Galois over R'. II. $R' \not\supseteqq Z$. Let $t \in Z \backslash R'$. Then $R'+R't$ properly contains R' and it is a two-sided R'-module. To be easily verified $[R'+R't \mid R']_l=2[R']$ and $R'+R't=R' \oplus R't=R$. For, if $R' \frown R't \neq 0$, then as $R' \frown R't$ is a two-sided R'-module contained in R' and $R't$, it has to coincide with $R't$ as well as R'. But this is a contradiction. Thus we obtain $t^2=a_1t+a_2$ for some a_i in R'. Since t and t^2 are in Z, this gives $(aa_1)t+aa_2=(a_1a)t+a_2a$ for each a in R'. Hence $aa_i=a_ia$ and so that a_i are in Z'. Since $R=R'+R't$ where t belongs to Z, it is clear that $Z'=R' \frown Z$. Hence a_i are in Z. We may replace t by $u=t-\frac{1}{2}a_1$ and obtain $u^2=c \in Z$ and $R=R' \oplus R'u$. For $p, q \in R'$, the mapping $p+qu \rightarrow p-qu$ is an automorphism of R whose set of invariants is R'. Moreover, there holds that $V(R') = V(R'Z)=V(R)=Z$. Hence R is Galois over R'.

Remark. In part II of the above proof, it is easily seen that R is the Kronecker product over Z' of R' and a quadratic extension of Z'.

References

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