5. *Some Trigonometrical Series. XVIII*

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1. Let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{1}$$

$$g(x) \sim \frac{a'_0}{2} + \sum_{n=1}^{\infty} (a'_n \cos nx + b'_n \sin nx). \tag{2}$$

The Parseval relation of $f(x)$ and $g(x)$ is

$$\frac{1}{\pi} \int_0^{2\pi} f(x) g(x) dx = \frac{a_0 a'_0}{4} + \sum_{n=1}^{\infty} (a_n a'_n + b_n b'_n). \tag{3}$$

The known conditions of $f(x)$ and $g(x)$ for the validity of (3) are as follows [1]:

(i) $f \varepsilon L^p$, $g \varepsilon L^{p'}$, $p>1$, $1/p+1/p'=1$.

(ii) $f \varepsilon L$, $g \varepsilon B$.

(iii) $f \varepsilon C$, $g \varepsilon S$.

In the case (i), the right side of (3) converges but in the cases (ii) and (iii), it is summable $(C, 1)$. It is known that, if

(ii′) $f \varepsilon L$, $g \varepsilon BV$, or (ii″) $f \varepsilon Z$, $g \varepsilon B$,

then the Parseval relation (3) holds and the right side is convergent.

We shall here prove that, in the cases (ii) and (iii) the right side of (3) converges under some additional conditions of $f(x)$ or $g(x)$.

On the other hand, let

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad g(x) \sim \sum_{n=-\infty}^{\infty} c'_n e^{inx}$$

and $f_\alpha(x)$ be the αth integral of $f(x)$. G. H. Hardy and J. E. Littlewood [2] proved that if

(iv) $f \varepsilon L^p$, $g \varepsilon L^q$, $p<2$, $q \leqq p'$ $(1/p+1/p'=1)$

then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f_\alpha(x) g(-x) dx = \sum_{n=-\infty}^{\infty} \frac{e^{-\frac{1}{2}\alpha\pi i \operatorname{sgn} n}}{|n|^\alpha} c_n c'_n \quad \text{for } \alpha = \frac{1}{p} + \frac{1}{q} - 1. \tag{4}$$

The case (iv) corresponds to the case (i) of (3). We shall prove (4) in the case corresponding to (ii).

For the proof of our theorem we use the following theorem due to one of us [3] (cf. [4]).

Theorem A. *If*

$$\frac{1}{\delta} \int_0^{\delta} |f(x+t) - f(x)| dt = o(1) \qquad (\delta \to 0) \tag{5}$$

for an x and

$$(6) \qquad \frac{1}{\delta}\int_0^\delta (f(y+t)-f(y-t))\,dt = o\left(1\Big/\log\frac{1}{\delta}\right) \quad (\delta\to 0)$$

uniformly for all y, *then the Fourier series of* $f(t)$ *converges to* $f(x)$ *at* x.

2. **Theorem 1.** *If* $g(t)$ *is integrable and* $f(t)$ *is a bounded measurable function such that*

$$(7) \qquad \int_0^h (f(x+u)-f(x-u))\,du = o\left(h\Big/\log\frac{1}{h}\right)$$

uniformly for all x *as* $h\to 0$, *then the Parseval relation* (3) *holds, where the right side converges.*

Proof. Let $s_N(t)$ be the Nth partial sum of the series (1), that is,

$$s_N(t)=\frac{a_0}{2}+\sum_{n=1}^N (a_n\cos nt+b_n\sin nt)=\frac{1}{\pi}\int_0^{2\pi} f(x)D_N(x-t)\,dx.$$

Then

$$\begin{aligned}\frac{a_0a_0'}{4}+\sum_{n=1}^N (a_na_n'+b_nb_n') &= \frac{1}{\pi}\int_0^{2\pi} g(t)s_N(t)\,dt\\ &=\frac{1}{\pi^2}\int_0^{2\pi} g(t)\,dt\int_0^{2\pi} f(x+t)D_N(x)\,dx\\ &=\frac{1}{\pi}\int_0^{2\pi} D_N(x)\,dx\cdot\frac{1}{\pi}\int_0^{2\pi} g(t)f(x+t)\,dt.\end{aligned}$$

Hence the right side is the Nth partial sum of the Fourier series of the function

$$(8) \qquad h(x)=\frac{1}{\pi}\int_0^{2\pi} g(t)f(x+t)\,dt$$

at $x=0$. In order to prove the theorem, it is sufficient to prove that $h(x)$ satisfies the conditions (5) (at $x=0$) and (6) in Theorem A.

Now

$$\begin{aligned}&\frac{1}{\delta}\int_0^\delta |h(u)-h(0)|\,du\\ &\quad=\frac{1}{\pi\delta}\int_0^\delta du\left|\int_0^{2\pi} g(t)[f(t+u)-f(t)]\,dt\right|\\ &\quad\leqq\frac{1}{\pi\delta}\int_0^{2\pi}|g(t)|\,dt\int_0^\delta |f(t+u)-f(t)|\,du\end{aligned}$$

which is $o(1)$ since $g\in L(0,2\pi)$ and the inner integral is uniformly $O(\delta)$ and is almost everywhere $o(\delta)$. We have also

$$\begin{aligned}&\frac{1}{\delta}\int_0^\delta (h(y+u)-h(y-u))\,du\\ &\quad=\frac{1}{\pi\delta}\int_0^\delta du\int_0^{2\pi} g(t)[f(y+u+t)-f(y+t-u)]\,dt\end{aligned}$$

$$=\frac{1}{\pi\delta}\int_0^{2\pi} g(t-y)dt\int_0^{\delta}[f(t+u)-f(t-u)]du$$

which is $o\left(1\Big/\log\frac{1}{\delta}\right)$ uniformly by $g\in L(0,2\pi)$ and the condition (7). Thus the theorem is proved.

Similarly we can prove the following

Theorem 2. *If $g(t)$ is bounded measurable and if $f(t)$ is an integrable function such that*

$$\int_0^{2\pi}dt\left|\frac{1}{\delta}\int_0^{\delta}[f(t+u)-f(t-u)]du\right|=o\left(1\Big/\log\frac{1}{\delta}\right),$$

then the Parseval relation (3) holds where the right side series converges.

3. **Theorem 3.** *Let $g(t)$ be of bounded variation and its Fourier-Stieltjes series be*

$$dg(t)\sim\frac{a_0'}{2}+\sum_{n=1}^{\infty}(a_n'\cos nt+b_n'\sin nt).$$

If $f(t)$ is a bounded (B) measurable function with Fourier series (1) such that

$$\int_0^{\delta}(f(x+u)-f(x-u))du=o\left(\delta\Big/\log\frac{1}{\delta}\right)\tag{9}$$

uniformly for all x, then the Parseval relation

$$\frac{1}{\pi}\int_0^{2\pi}f(t)dg(t)=\frac{a_0a_0'}{4}+\sum_{n=1}^{\infty}(a_na_n'+b_nb_n')\tag{10}$$

holds, where the right side converges.

Proof is similar as in Theorem 1.

Theorem 4. *Let $g(t)$ be a continuous function with Fourier series (2). If $f(x)$ is of bounded variation and its Fourier-Stieltjes series is*

$$df(t)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)$$

and further if the total variation of

$$F(t)=\frac{1}{\delta}\int_0^{\delta}(f(t+u)-f(t-u))du$$

is of order $o\left(1\Big/\log\frac{1}{\delta}\right)$, then the Parseval relation

$$\frac{1}{\pi}\int_0^{2\pi}g(t)df(t)=\frac{a_0a_0'}{4}+\sum_{n=1}^{\infty}(a_na_n'+b_nb_n')$$

holds, where the right side converges.

Proof. As in the proof of Theorem 1, it is sufficient to prove that the function

$$h(x)=\frac{1}{\pi}\int_0^{2\pi}g(t)d_tf(x+t)=-\frac{1}{\pi}\int_0^{2\pi}g(t-x)df(t)$$

satisfies the conditions (5) at $x=0$ and (6).

Now

$$\frac{1}{\delta}\int_0^\delta |h(u)-h(0)|\,du=\frac{1}{\pi\delta}\int_0^\delta du\left|\int_0^{2\pi}[g(t-u)-g(t)]df(t)\right|$$
$$\leqq\frac{1}{\pi}\int_0^{2\pi}|df(t)|\cdot\frac{1}{\delta}\int_0^\delta|g(t-u)-g(t)|\,du=o(1)$$

by the continuity of $g(t)$, and

$$\frac{1}{\delta}\int_0^\delta(h(y+u)-h(y-u))du$$
$$=\frac{1}{\pi\delta}\int_0^\delta du\int_0^{2\pi}g(t)d_t[f(y+t+u)-f(y+t-u)]$$
$$=\frac{1}{\pi}\int_0^{2\pi}g(t-y)d_t\left(\frac{1}{\delta}\int_0^\delta(f(t+u)-f(t-u))du\right).$$

By the condition of the theorem

$$\int_0^{2\pi}\left|d_t\left(\frac{1}{\delta}\int_0^\delta[f(t+u)-f(t-u)]du\right)\right|=o\left(1\Big/\log\frac{1}{h}\right).$$

Thus we get the theorem.

4. **Theorem 5.** *If $g(t)\,\varepsilon\,L(0,2\pi)$ and $f(t)\,\varepsilon\,L^p(0,2\pi)$ $(p>1)$, then the Parseval relation (4) holds for $\alpha>1/p$, where the right side series converges.*

Proof. If $\alpha\geqq 1+1/p$, then the theorem is evident. Hence we can suppose that $1+1/p>\alpha>1/p$, and then $f_\alpha(x)\,\varepsilon\,\mathrm{Lip}\,(\alpha-1/p)$ by the Hardy-Littlewood theorem. By Theorem 1, we get the required result.

References

[1] A. Zygmund: Trigonometrical series, Warszawa (1935).
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