58. Note on the Lebesgue Property in Uniform Spaces. II

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Let E be a separated uniform space. An open covering $\mathfrak{S}$ of E is said to have the *Lebesgue property* if there exists a surrounding V of E such that, for each point $x \in E$, we can find a member O of the covering $\mathfrak{S}$ containing the set $V(x)$. If every covering of E consisting of a countable number of open sets (such a covering is called a *countable open covering*) has the Lebesgue property, the space E is said to have the *countable Lebesgue property.* All uniform spaces to be considered in what follows will be assumed separated.

In a previous note [4], it has been shown among other things that a uniformisable Hausdorff space possesses a compatible uniform structure for which every open covering of the space has the Lebesgue property if and only if it is paracompact, and the corresponding proposition to the finite Lebesgue property.[1]

A purpose of this note is to prove an analogous proposition for the countable Lebesgue property, and to give a characterization of a uniform space having the countable Lebesgue property.

First of all we can easily obtain the following

THEOREM 1. *Let E be a uniformisable Hausdorff topological space, then E possesses a uniform structure compatible with the topology of E for which E has the countable Lebesgue property if and only if E is countably paracompact*[2] *and normal.*

It is known[3] that a uniform space having the countable Lebesgue property is countably paracompact and normal, and so we have only to prove the if part of the theorem. Now, by a theorem of C. H. Dowker [1] and a theorem of K. Morita [5], we can conclude that every countable open covering of a countably paracompact normal space is normal[4] and the normal sequence consists of countable open coverings.

Therefore, the family of all countable open coverings of E forms

1) A separated uniform space is said to have the *finite Lebesgue property* if every covering consisting of a finite number of open sets has the Lebesgue property.

2) A Hausdorff topological space E is *countably paracompact* if every countable open covering of E has a locally finite refinement. Cf. C. H. Dowker [1].

3) Cf. S. Kasahara [3].

4) A sequence $\{\mathfrak{S}_n\}$ of coverings of a space is called a *normal sequence* if, for each n, the covering $\mathfrak{S}_{n+1}$ is a star refinement of $\mathfrak{S}_n$. A covering $\mathfrak{S}$ is normal if a normal sequence $\{\mathfrak{S}_n\}$ exists such that $\mathfrak{S}_1$ is a refinement of $\mathfrak{S}$. Cf. J. W. Tukey [7].

a basis of a uniformity compatible with the topology of E. The uniform structure corresponding to this uniformity is called the *E-structure*. Explicity, E has the countable Lebesgue property for any compatible uniform structure finer than the E-structure.

Remark. It is clear that a countably paracompact normal space is countably compact if and only if the E-structure is equivalent to the F-structure.[5] On the other hand, in a normal space, the F-structure is equivalent to the universal structure[6] if and only if for every compatible uniform structure it is precompact. But, this means by a theorem of P. Samuel [6] that the normal space is countably compact. Hence

1°. For a countably paracompact normal space E, the following conditions are equivalent:

(1) E is countably compact;
(2) the E-structure is equivalent to the F-structure;
(3) the F-structure is universal.

2°. For a paracompact space E, the following conditions are equivalent:

(1) E is compact;
(2) the E-structure is equivalent to the F-structure;
(3) the F-structure is universal;
(4) E has a unique uniform structure.

Obviously, a paracompact space has the Lindelöf property if, and only if, the E-structure is universal.

In [3] we have shown that, in a uniform space, every open covering having the Lebesgue property is shrinkable. However, more precisely, we can say that an open covering of a uniform space has the Lebesgue property if and only if it is, say *uniformly shrinkable*, that is

THEOREM 2. *An open covering $\{O_\alpha\}_{\alpha\in A}$ of a uniform space E has the Lebesgue property if and only if there exists a surrounding V of E such that the collection $\{V(O_\alpha^c)^c\}_{\alpha\in A}$ is a covering of E* (where c denotes the complement operator).

Let V be the symmetric surrounding ensured by the Lebesgue property of the covering $\{O_\alpha\}_{\alpha\in A}$, then we have

5) The uniform structure corresponding to the uniformity consisting of all finite open coverings of a normal space is called the *F-structure* of the space. Cf. S. Kasahara and K. Kasahara [4].

6) A uniform structure of a uniformisable space E is *universal* if it is finer than any compatible uniform structure of E. The foot-note 4 of [3, p. 130] is inexact, and it should be read as follows: E is countably compact if E has the finite Lebesgue property and it is pseudo-compact.

$$\bigcap_{\alpha\in A} V(O^c_\alpha)=0,$$

since otherwise there is an $x \in E$ such that $V(x) \frown O^c_\alpha \neq 0$ for any $\alpha \in A$; this states the only if part of the theorem. It is also easy to show the validity of the if part, and we omit the proof.

Using this theorem we shall now prove the following

THEOREM 3. *A uniform space E has the countable Lebesgue property if and only if E is countably paracompact normal and every continuous mapping of E into any separable uniform space is uniformly continuous.*

To prove the only if part of our theorem, let us consider a separable uniform space F and denote by I a countable dense subset of F. For any surrounding U' of F, we choose a symmetric surrounding U of F such that $\overset{2}{U} \subset U'$. Let us denote by V the surrounding of E ensured by the Lebesgue property of the countable open covering $\{f^{-1}(U(i))\}_{i\in I}$, then we have

$$(f(x), f(y)) \in U' \qquad \text{for any } (x, y) \in V.$$

For, if $(x, y) \in V$, there is an $i \in I$ such that $V(x) \subset f^{-1}(U(i))$, and so we have $(f(x), i), (f(y), i) \in U$.

We shall now proceed to prove the if part of the theorem. Let F be the set of all sequences of real numbers a_i such that $0 \leq a_i \leq 1$ and that all but a finite number of the a_i's are equal to 0. For any two points $a=(a_i)$ and $b=(b_i)$ of F, put

$$d(a, b)=\sum_{i=1}^{\infty} | a_i - b_i |,$$

then, as can be easily seen, the set F becomes a separable metric space with respect to this metric d. Now, let $\{O'_i\}$ be an arbitrary countable open covering of E, then the space E being countably paracompact normal, there exists a locally finite countable refinement $\{O_i\}$ of $\{O'_i\}$. To obtain the desired conclusion, it will suffice to show that the countable open covering $\{O_i\}$ has the Lebesgue property. Then, by a theorem of C. H. Dowker [1], the covering $\{O_i\}$ is shrinkable, that is to say, we can take a countable open covering $\{U_i\}$ of E such that $\overline{U}_i \subset O_i$ for any $i=1, 2, \cdots$. Since the space E is normal, there exists, for any i, a real valued continuous function f_i such that

$$0 \leq f_i(x) \leq 1 \quad \text{for any } x \in E, \text{ and } \quad f_i(x)=\begin{cases} 1 & \text{for any } x \in \overline{U}_i; \\ 0 & \text{for any } x \bar{\in} O_i. \end{cases}$$

Let us consider a mapping $f(x)=(f_i(x))$ of E into the above-defined separable metric space F, then it is clear that the mapping f is well defined and is continuous since the covering $\{O_i\}$ is locally finite. Therefore, by the assumption, f is uniformly continuous, and so we can find a symmetric surrounding V such that

$$(*) \qquad d(f(x), f(y)) \leq \frac{1}{2} \qquad \text{whenever } (x, y) \in V.$$

If the total intersection of the family $\{V(O_i^c)\}$ is non-empty, then there exists an $x \in E$ belonging to all $V(O_i^c)$. On the other hand, the point x is contained in some $\overline{U}_i$, and for this positive integer i, we can take therefore a point $y \in O_i$ such that $(x, y) \in V$. But then, since $f_i(x)=1$ and $f_i(y)=0$, we have $d(f(x), f(x)) \geq 1$, which contradicts the relation $(*)$. Accordingly, we must have $\bigcap_{i=1}^{\infty} V(O_i^c)=0$, and thus the countable open covering $\{O_i\}$ has the Lebesgue property in view of Theorem 2.

In the remainder of this note, we will concern the completion of uniform spaces of some types.

THEOREM 4. *Let E be a uniform space having the countable Lebesgue property. Then the completion $\hat{E}$ of E has the countable Lebesgue property if and only if $\hat{E}$ is countably paracompact and normal.*

In view of Theorems 1 and 3, it will suffice to prove that every continuous mapping f of $\hat{E}$ into a separable uniform space F is uniformly continuous. Now let $\hat{F}$ be the completion of F, then since E has the countable Lebesgue property, the restricted mapping $f \mid E$ of f to E is uniformly continuous as a mapping of E into $\hat{F}$. Hence, $f \mid E$ may be extended uniform continuously to $\hat{E}$ and the extension coincides with f as can be readily seen. Thus the mapping f is uniformly continuous.

THEOREM 5 *Let E be a uniform space. If the completion $\hat{E}$ of E has the countable Lebesgue property and for any sequence of closed sets F of E we have*

$$(**) \qquad \overline{\bigcap_{i=1}^{\infty} F_i} = \bigcap_{i=1}^{\infty} \overline{F}_i,$$

where the closure is taken in $\hat{E}$, then E has the countable Lebesgue property and $\hat{E}$ is countably paracompact normal: and the converse is also true.

First, we proceed to show that the space E has the countable Lebesgue property. Let us consider an arbitrary sequence of closed sets F_i of E with empty total intersection. Then, by the assumption, we have $\bigcap_{i=1}^{\infty} \overline{F}_i=0$. And so, since $\hat{E}$ has the countable Lebesgue property, for a suitable surrounding V of $\hat{E}$, we have $\bigcap_{i=1}^{\infty} V(\overline{F}_i)=0.$[7]

7) Cf. Theorem 2.

Let $U=V\frown(E\times E)$, then $U(F_i)$ is contained in $V(\overline{F}_i)$ and we have thus $\bigcap_{i=1}^{\infty} U(F_i)=0$, which states that the space E has the countable Lebesgue property.

In order to complete the proof of the theorem, it remains only to prove the following

THEOREM 6. *If a uniform space E has the countable Lebesgue property, then for any sequence of closed sets F_i of E, we have the relation* (**).

To prove this, it is sufficient to show that

$$(\dagger)\qquad \text{if } \bigcap_{i=1}^{\infty} F_i=0, \text{ then } \bigcap_{i=1}^{\infty} \overline{F}_i=0.$$

In fact, we have to prove that $\overline{\bigcap_{i=1}^{\infty} F_i}\supset \bigcap_{i=1}^{\infty} \overline{F}_i$, but if there is a point x in $\bigcap_{i=1}^{\infty} \overline{F}_i-\overline{\bigcap_{i=1}^{\infty} F_i}$, we can find a neighbourhood U of x in $\hat{E}$ such that $U\frown\bigcap_{i=1}^{\infty} F_i=0$. Put $H_i=\overline{U}\frown F_i$, then the set $\bigcap_{i=1}^{\infty} \overline{H}_i$ contains the point x and $\{H_i\}$ is a sequence of closed sets of E with empty total intersection. Thus if the assertion ($\dagger$) has been proved for any sequence of closed sets F_i of E, a contradiction occurs and the proofs are completed. However, this may be easily obtained from the lemma below.

LEMMA. *Let F be a uniform space, and E a dense subspace of F with the induced uniform structure. If an open covering $\{O_\alpha\}$ of E has the Lebesgue property, then the collection $\{F-\overline{E-O_\alpha}\}$ is an open covering of F having the Lebesgue property.*

Suppose that the open covering $\{O_\alpha\}$ of E has the Lebesgue property; then there exists a symmetric open surrounding V of F such that, for each point $y\in E$, the set $U(y)$ is contained in some O_α, where $U=V\frown(E\times E)$. Let W be a symmetric surrounding of F with $\overset{2}{W}\subset V$, then for any point $x\in F$, we can take a point $y\in E$ belonging to the set $W(x)$. Since $E-U(y)=(F-V(y))\frown E$, we have $F-\overline{E-U(y)}\supset V(y)$. But the set $U(y)$ is contained in some O_α, and $V(y)$ is consequently in $F-\overline{E-O_\alpha}$. On the other hand, the set $W(x)$ is contained in $V(y)$ because for any $z\in W(x)$ the pair (z,y) belongs to V. Therefore, we have $W(x)\subset F-\overline{E-O^\alpha}$, proving the lemma.

By a similar method, it is possible to prove the following analogous theorem for the finite Lebesgue property, and we shall omit the proof.

THEOREM 7. *Let E be a uniform space. If E has the finite Lebesgue property and its completion $\hat{E}$ is normal, then $\hat{E}$ has the* *finite Lebesgue property and for any closed sets F_1, F_2 of E we have*

$$\overline{F_1 \frown F_2} = \overline{F_1} \frown \overline{F_2}$$

where the closure is taken in $\hat{E}$. The converse is also true.

We note that the theorems concerned with the finite Lebesgue property of type of Theorems 4 and 6 have been proved by K. Iséki in his note [2].

References

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