54. Evans-Selberg's Theorem on Abstract Riemann Surfaces with Positive Boundaries. II

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Our $N_{V_m(p)}(p,q)$ is increasing with respect to m. We define the value of $N(z,q)$ at a minimal point p by $\lim_{m\to M'} N_{V_m(p)}(p,q)$ denoted by $N(p,q)$. If p or q belongs to R, this definition is equivalent to that defined before.

If $V_m(p)$ is not regular, we define $N_{V_m(p)}(p,q)$ by $\lim_{m'\to m} N_{V_{m'}(p)}(p,q)$, where $m'<m$ and $V_{m'}(p)$ is regular. In the case when $V_m(p)$ is regular, it is proved that $\lim_{m'\to m} N_{V_{m'}(p)}(p,q)=N_{V_m(p)}(p,q)$, hence we can define $N_{V_m(p)}(p,q)$ for every $m<\sup_{z\in R} N(z,p)=M'$. As in case of a Riemann surface with a null-boundary, we can prove the following

Theorem 10. 1) *$N(z,q)$ $(q\in\overline{R})$ is δ-lower semicontinuous in $R+B_1$.*

2) *$N(z,q)$ is superharmonic in weak sense at every point of $R+B_1$.*

3) *If p and q are in $R+B_1$, then $N(p,q)=N(q,p)$.*

Till now $N(z,q)$ $(q\in\overline{R})$ is defined only on $R+B_1$. Next we define $N(z,q)$ at points belonging to B_0. If $p\in B_0$, $N(z,p)=\int_{B_1} N(z,p_\alpha)d\mu(p_\alpha)$ $(p_\alpha\in B_1)$ by Theorem 8. Although the uniqueness of this mass distribution is not proved by the present author, the value of $N(z,q)$ in $R+B_1$ is uniquely determined. On the other hand, by 3), for $q\in B_1$, $N(p_\alpha,q)=N(q,p_\alpha)$. Hence it is quite natural to define the value of $N(z,q)$ at $p\in B_0$ by $\int N(p_\alpha,q)d\mu(p_\alpha)$. Evidently by 3), in such definition, we have $N(q,p)=N(p,q)$, where the term of the right hand side does not depend on a particular distribution but on the behaviour of $N(z,q)$, because $N(p,q)=\lim_{m\to M'} N_{V_m(p)}(p,q)$ and $N_{V_m(p)}(p,q)$ is defined by the value of $N(z,q)$ on $\partial V_m(p)$. As for the behaviour of $N(z,q)$ $(q\in\overline{R})$, we have the following

Theorem 11. 1) *If $q\in R+B_1$, then $N(p,q)=N(q,p)$ for $p\in\overline{R}$.*

2) *If $q\in\overline{R}$ and $p\in R+B_1$, then $N(p,q)=\int N(p,q_\alpha)d\mu(q_\alpha)$, where $N(z,q)=\int N(z,q_\alpha)d\mu(q_\alpha)$.*

3) *$N(z,q)$ $(q\in\overline{R})$ is δ-lower semicontinuous in $\overline{R}$.*

1′) For every p and q belonging to $\overline{R}$, $N(p,q)=N(q,p)$.

7. *Potentials on $\overline{R}$.* In the sequel, we shall study the mass distributions on $\overline{R}$. We have seen that $N(z,p)$ has the essential properties of logarithmic potential: lower semicontinuity in $\overline{R}$, symmetricity and superharmonicity in $R+B_1$. But there exists the fatal difference between our case and space, that is, the real mass distribution can be defined only on $R+B_1$, i.e. the distribution on B_0 is superficial and it can be replaced, by Theorem 8, by that on B_1 where $N(z,p)$ is superharmonic. Therefore only subsets $R+B_1$ of $\overline{R}$ can be a kernel of mass distribution. Hence it is easy to construct the potential theory on $\overline{R}$.

The energy integral $I(\mu)$ of a mass distribution μ on a δ-closed subset of $R+B_1$ defined as in space

$$I(\mu)=\int\int N(q,p)d\mu(p)d\mu(q)$$

and the $\overset{*}{\text{c}}$apacity is defined as usual. In § 1, we defined capacity of F, we must study the relation between two capacities. At first, we have, if $\text{Cap}(F)>0$, $\overset{*}{\text{Cap}}(F)>0$. Now we have the following

Theorem 12. Let F be a δ-closed subset of $R+B_1$ of $\overset{*}{\text{c}}$apacity positive. Then there exists a unit mass distribution on F whose energy is minimal and whose potential $U(z)$ has the following properties:

1) $U(z)$ is a constant C on the kernel of this distribution.

2) $U(z)=C$ on E except possibly a set of $\overset{*}{\text{c}}$apacity zero.

3) $U(z)=U_F(z)$.

4) $U(z)=C\omega_F(z)$, where $\omega_F(z)$ is the equilibrium potential of F.

By 2) of this theorem and by 2) of Theorem 5, we have the following

Corollary. $\text{Cap}(F)=\overset{*}{\text{Cap}}(F)$.

Transfinite diameter. Since $N(z,p)$ $(p\in\overline{R})$ is δ-lower semicontinuous in $\overline{R}$, the transfinite diameter of a δ-closed subset A of $\overline{R}$ is defined as follows:

$$\frac{1}{D_A}=\lim_{n=\infty}\Bigl(\min\Bigl(\frac{1}{{}_nC_2}\ (\sum_{\substack{i\neq j\\ i,j=1\\ p_i,p_j\in A}}^{n,n}\ (p_i,p_j))\Bigr)\Bigr).$$

Then as in the case of R^* with a null-boundary, we have the following

Theorem 13. If $D_A=0$ for a δ-closed subset A of $\overline{R}$, then there exists a superharmonic function $U(z)$ in $\overline{R}$ such that $U(z)=0$ on ∂R_0, $\int_{\partial}\frac{\partial U(z)}{\partial n}ds=2\pi$ and $U(z)=\infty$ at every point of A.

For a δ-closed subset A of $R+B_1$, it can be proved as in space

$D_A=\frac{1}{I(\mu)}$, where $I(\mu)$ is the energy of the equilibrium potential of A. Hence we have the following

Theorem 14 (*Extension of Evans-Selberg's theorem*). *Let A be a δ-closed subset of $R+B_1$, of capacity zero. Then there exists a unit mass distribution on A whose potential satisfies the following properties*:

1) $U(z)=0$ *on* ∂R_0.

2) $U(z)=\infty$ *at every point of* A.

3) $U(z)=U_A(z)$.

4) $\int\limits_{\partial R_0}\frac{\partial U(z)}{\partial n}ds=\int\limits_{C_r}\frac{\partial U(z)}{\partial n}ds$, *for the niveau curve* C_r *of* $U(z)$ *with* $r\notin E$, *where E is a set in the interval* $[0,\infty]$ *such that* $\operatorname{mes}E=0$.

In general cases we can not omit the condition that A is a subset of $R+B_1$. The reason is as follows: there may exist a set B_0 which is an F_σ and of capacity zero and any mass can not be distributed on B_0, in other words, B_0 has behaviour like an empty set in space for mass distribution though B_0 is not empty.

The value of $U(z)$ at a point $p\in B_0$ is given as follows: since $N(z,p)=\int\limits_{B_1}N(z,p_\alpha)d\mu(p_\alpha)(p_\alpha\in B_1)$, $U(p)=\int U(p_\alpha)d\mu(p_\alpha)$. Therefore $U(z)$ may be infinite at larger set A' than A.