75. Notes on Topological Spaces. III. On Space of Maximal Ideals of Semiring

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S. Bourne [1] considered the Jacobson radical of a semiring and recently W. Slowikowski and W. Zawadowski [6] developed the general theory and space of maximal ideals of a positive semiring. R. S. Pierce [4] considered a topological space obtained from a semiring. A. A. Monteiro [3] wrote an excellent report on representation theory of lattices.

Definition 1. A *semiring* A is an algebra with two binary operations, addition (written $+$) which is associative, and multiplication which is associative, and satisfies the distributive law

$$a(b+c)=ab+ac, \quad (b+c)a=ba+ca.$$

In this paper, we suppose that A has the further properties:

1) There are two elements $0, 1$ such that

$$x+0=x, \qquad x\cdot 1=x$$

for every x of A.

2) Two operations, addition and multiplication, are commutative.

Definition 2. A non-empty proper subset I of A is called an *ideal*, if

(1) $\quad a, b \in I$ implies $a+b \in I$,

(2) $\quad a \in I, x \in A$ implies $ax \in I$.

W. Slowikowski and W. Zawadowski [6] proved that *every ideal is contained in a maximal ideal.* An ideal is maximal if there is no ideal containing properly it.

Let $\mathfrak{M}$ be the set of all maximal ideals in a semiring A. We shall define two topologies on $\mathfrak{M}$.

For every x of A, we denote by Δ_x the set of all maximal ideals containing x, and by Γ_x the set $\mathfrak{M}-\Delta_x$, i.e. the set of all maximal ideals not containing x. Let I be an ideal of A, we denote by Δ_I the set of all maximal ideals containing I.

We shall choose the family $\{\Delta_x \mid x \in A\}$ as a subbase for open sets of $\mathfrak{M}$. We shall refer to the resulting topology on $\mathfrak{M}$ as Δ-topology (in symbol, $\mathfrak{M}_\Delta$). Similarly, we shall take the family $\{\Gamma_x \mid x \in A\}$ as a subbase for open sets of $\mathfrak{M}$ (in symbol, $\mathfrak{M}_\Gamma$). These two topologies for normed ring or general commutative ring were considered by I. Gelfand and G. Silov [2] or P. Samuel [5].

Let M_1, M_2 be two distinct elements of $\mathfrak{M}_\Delta$. Then we have $M_1+M_2=A$. Therefore there are a, b such that $a+b=1$ and $a \in M_1$,

$b \in M_2$, so we have $\Delta_a \ni M_1$, $\Delta_b \ni M_2$ and $\Delta_a \frown \Delta_b = 0$. Hence

Theorem 1. The topological space $\mathfrak{M}_\Delta$ *is a* T_2*-space.*

Let M be an element of $\mathfrak{M}_\Gamma$, and $M \neq M_1 \in \mathfrak{M}_\Gamma$, then there is an element a such that $a \in M_1$ and $a \notin M$. Therefore $\Gamma_a \not\ni M_1$ and $\bigcap_{x \notin M} \Gamma_x \not\ni M_1$. This implies $M = \bigcap_{x \notin M} \Gamma_x$. Hence we have the following

Theorem 2. The topological space $\mathfrak{M}_\Gamma$ *is a* T_1*-space.*

Let I be an ideal of A and $\{a_\lambda\}$ a generator of I, then we have

$$\Delta_I = \bigcap_\lambda \Delta_{a_\lambda}.$$

Therefore, the closed sets for the topological space $\mathfrak{M}_\Gamma$ have the form $\Delta_{I_1} \smile \Delta_{I_2} \smile \cdots \smile \Delta_{I_n}$, where I_i are ideals of A.

Let $I = \bigcap_{i=1}^n I_i$, if $\Delta_{I_i} \ni M$ for some i, then $M \supset I_i$ and $M \supset I$. This implies $\Delta_I \ni M$ and we have $\bigcup_{i=1}^n \Delta_{I_i} \subset \Delta_I$. Suppose that there is a maximal ideal M such that $M \in \Delta_I - \bigcup_{i=1}^n \Delta_{I_i}$, then $M \in \Delta_I$ and $M \notin \bigcup_{i=1}^n \Delta_{I_i}$. Hence $M \supset I$ and M does not contain every I_i $(i=1, 2, \cdots, n)$. Therefore, since M is a maximal ideal, there are elements $a_i \in I_i$ and $m_i \in M$ such that

$$a_i + m_i = 1 \quad (i=1, 2, \cdots, n).$$

Thus, we have

$$1 = a_1 a_2 \cdots a_n + m, \quad m \in M$$

and $a_1 a_2 \cdots a_n \in I$. This implies $I + M = A$. Hence, by $I \subset M$, we have $M = A$, which is a contradiction. This shows the following relation:

$$\bigcup_{i=1}^n \Delta_{I_i} = \Delta_I$$

and we have the following

Theorem 3. The closed sets for $\mathfrak{M}_\Gamma$ *are expressed by sets* Δ_I*, where* I *is an ideal of* A.

By Theorem 3, we shall show the following

Theorem 4. The space $\mathfrak{M}_\Gamma$ *is a compact* T_1*-space.*

To prove it, let $\{\Delta_{I_\lambda}\}$ be a family of closed sets in $\mathfrak{M}_\Gamma$ with the finite intersection property, where I_λ are ideals in A. Therefore, any finite family of I_λ does not generate the semiring A. Hence the ideal I generated by $\{I_\lambda\}$ does not contain the unit 1 of A. This shows that I is contained in a maximal ideal M. Hence

$$\bigcap_\lambda \Delta_{I_\lambda} \ni M.$$

Therefore, since $\bigcap_\lambda \Delta_{I_\lambda}$ is non-empty, $\mathfrak{M}_\Gamma$ is a compact space.

Example. Let A be the semiring of non-negative integers with ordinary addition and multiplication. An ideal I of A is maximal, if and only if, there is a prime number p such that $I = (p)$. As closed set of $\mathfrak{M}_\Gamma$ is finite, any distinct two elements of $\mathfrak{M}_\Gamma$ can not separate by disjoint open sets. Hence $\mathfrak{M}$ for Γ-topology is not a T_2-space.

Following W. Slowikowski and W. Zawadowski [6], we shall define positive semirings.

Definition 3. A semiring A is *positive*, if, for every a of A, $1+a$ has an inverse.

Let A be a positive semiring, then, for any element a of A, there is an element b such that $ab+b=1$, i.e. $(a)+(b)=A$. This means that, for every element a of a positive semiring A, A contains at least one element b such that A is generated by a and b. Hence any maximal ideal M containing b does not contain a. Consequently $\Delta_b \subset \Gamma_a$. Hence we have

Lemma 1. Every open set of $\mathfrak{M}_\Gamma$ *for a positive semiring contains an open set of* $\mathfrak{M}_\Delta$.

Any set Γ_a is a closed set for $\mathfrak{M}_\Delta$. If Γ_a is a closed set for $\mathfrak{M}_\Gamma$, then there is an ideal I of A such that $\Gamma_a = \Delta_I$ by Theorem 3. If $(a)+I \neq A$, then there is a maximal ideal M containing $(a)+I$, and $\Gamma_a \not\ni M$ and $M \in \Delta_I$, therefore this implies $\Gamma_a \neq \Delta_I$. Hence we have $(a)+I=A$, so there are such elements $x \in A$ and $b \in I$ that $ax+b=1$. This shows that any maximal ideal containing b does not contain a. Hence $\Delta_b \subset \Gamma_a$. Clearly, $\Delta_I \subset \Delta_b$. Therefore $\Delta_b = \Gamma_a$ by $\Gamma_a = \Delta_b$.

Lemma 2. If Γ_a *is closed for* $\mathfrak{M}_\Gamma$ *of a positive semiring, then there is an element* b *such that* $\Delta_b = \Gamma_a$.

Conversely, we have easily the following

Lemma 3. If, for any element a *of* A, *there is an element* b *such that* $\Gamma_a = \Delta_b$, *then* Γ*-topology and* Δ*-topology on* $\mathfrak{M}$ *coincide.*

Hence we have the following

Theorem 5. Γ*-topology and* Δ*-topology for* $\mathfrak{M}$ *of a positive semiring* A *coincide, if and only if, for every* a *of* A, *there is an element* b *of* A *such that maximal ideals not containing* a *are same of the family of maximal ideals containing* b.

Definition 4. If for every two maximal ideals M, N in a semiring A, there are two elements $x \notin M, y \notin N$ such that xy is contained in the intersection of all maximal ideals of A, A is called *normal*.

It is known that A *is normal, if and only if* $\mathfrak{M}$ *is a normal space* (see W. Slowikowski and W. Zawadowski [6]).

Therefore we have

Theorem 6. If, for any element a *of* A, *there is an element* b *such that* $\Gamma_a = \Delta_b$, *then* A *is normal.*

Theorem 7. If any Γ_a *is closed of* $\mathfrak{M}_\Gamma$ *of a positive semiring* A, *then* A *is normal.*

In our later paper, we shall investigate the ideal structure of semiring and general theory of topological semiring.

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Correction to Yasue Miyanaga: "A Note on Banach Algebras" (Proc. Japan Acad., 32, No. 3, 176 (1956))

Page 176, line 6, for "all $x \in R$" read "regular elements $x \in R$".