73. *Local Properties of Topological Spaces*

By Keiô NAGAMI
Department of Mathematics, Ehime University
(Comm. by K. KUNUGI, M.J.A., May 15, 1956)

1. **Introduction.** We say that a topological space R has a certain property P of topological spaces locally if every point of R has a neighborhood with a property P and that P is a *local property* of R. If R is covered by a normal open covering in the sense of J. W. Tukey [5] each element of which has a property P, we say that R has a property P uniformly locally and that P is a *uniform local property* of R. If R itself has a property P, we say that P is a *global property* of R. Any global property P of R is of course a (uniform) local property and conversely there are several important properties P such that if R has P (uniformly) locally R has it globally. It is natural to raise the question:

Under what conditions is a (uniform) local property P of R a global property?

E. Michael [1] has shown that the axiomatic treatment can be applied to this problem, introducing the concepts such as G- and F-hereditary properties. The purpose of this paper is to show that the concepts of CG- or CF-hereditary property P (Definitions 1 and 2 below) can also be applied to this problem and to apply this results to several special cases.

2. **General cases.** A collection $\{S_\alpha\}$ of subsets of a topological space R is called *discrete* if $\{\overline{S}_\alpha\}$ is locally finite and mutually disjoint.

Definition 1. Let P be a property of topological spaces and R be a topological space. We say that P is a *CG-hereditary property* of R if the following conditions are satisfied:

(G1) If S is a subset of R with a property P, every elementary open[1] subset of S has also a property P.

(G2) If $\{S_\alpha\}$ is a discrete collection of subsets of R each element of which has a property P, $\cup S_\alpha$ has also a property P.

(G3) If $\{S_i\}$ is a countable locally finite collection of elementary open sets of R each element of which has a property P, $\cup S_i$ has also a property P.

A covering $\mathfrak{U}=\{U_\alpha; \alpha \in \cup A_i\}$ of a topological space R is called a strong screen if every $\mathfrak{U}_i=\{U_\alpha; \alpha \in A_i\}$ is discrete and $\{\bigcup_{\alpha\in A_i} U_\alpha; i=1, 2, \cdots\}$ is locally finite.

1) A subset S of R is called elementary open if S is of the form $\{x; f(x)>0\}$, where f is continuous.

Lemma 1. *Every normal open covering $\mathfrak{B}$ of a topological space R can be refined by a strong screen each element of which is an elementary open set.*

Combining [4] with [5, pp. 50–51], we get at once the lemma.

Theorem 1. *If a CG-hereditary property P is a uniform local property of a space R, P is a global property of R.*

Theorem 2. *If a CG-hereditary property P is a local property of a fully normal space R, P is a global property of R.*

Definition 2. Let P be a property of topological spaces and R be a topological space. We say that P is a *CF-hereditary property* of R if the following conditions are satisfied:

(F1) If S is a subset of R and has a property P, every elementary closed[2] subset of S has a property P.

(F2) If $\{S_\alpha\}$ is a discrete collection of closed subsets of R each element of which has a property P, $\smile S_\alpha$ has also a property P.

(F3) If $\{S_i\}$ is a countable locally finite collection of elementary closed subsets of R each element of which has a property P, $\smile S_i$ has also a property P.

Analogously to the preceding theorems we get the following theorems which are verified by the analogous ways.

Theorem 3. *If a CF-hereditary property P is a uniform local property of a space R, P is a global property of R.*

Theorem 4. *If a CF-hereditary property P is a local property of a fully normal space R, P is a global property of R.*

3. Applications. We shall give some applications of § 2.

Theorem 5. *Let S be a subset of a topological space R. If R is covered by a normal open covering $\{U_\alpha\}$ such that each $U_\alpha \frown S$ is respectively a Borel set of an additive or a multiplicative class $\leqq \beta$ (>0), then S is a Borel set of an additive or a multiplicative class $\leqq \beta$.*

This is an immediate consequence of the fact that the property of being respectively a Borel set of R of an additive or a multiplicative class $\leqq \beta$ is a *CG*- or a *CF*-hereditary property of R. Analogously to the above we get the following

Theorem 6. *Let S be a subset of a topological space R. If R is covered by a normal open covering $\{U_\alpha\}$ such that each $U_\alpha \frown S$ is an analytic set in R, then S itself is an analytic set in R.*

With the aid of Theorems 2 and 4, we get the following corollary, since every perfectly normal, fully normal space is hereditarily fully normal.

Corollary 1. *Let S be a subset of a perfectly normal, fully normal space R. If every point of S has a neighborhood U (in R) such that $U \frown S$ is an analytic set (in R), then S is an analytic set.*

---

2) A subset S of R is called elementary closed if $R-S$ is elementary open.

If every point of S has a neighborhood U (in R) such that $U \frown S$ is respectively a Borel set of an additive or a multiplicative class $\leqq \beta$, then S is a Borel set of an additive or a multiplicative class $\leqq \beta$.

This is a generalization of D. Montgomery's theorems [2, Theorems 1 and 2]. It is to be noted that if we replace "Borel" with "Baire" in all the preceding propositions, the propositions thus obtained also hold. With the aid of [3, Theorem 6], we get the following

Corollary 2. *Let R be a uniformly locally countably compact (sequentially compact) space. If the family of Borel sets of R coincides with the family of Baire sets, then R is perfectly normal.*

Theorem 7. *If a topological space R is respectively uniformly locally completely regular, uniformly locally normal, uniformly locally completely normal, uniformly locally perfectly normal, uniformly locally collectionwise normal, uniformly locally paracompact, uniformly locally hereditarily paracompact or uniformly locally metrizable, then all the words "uniformly locally" can be dropped.*

Some cases of the theorem have been explicitly or implicitly proved by several authors.

Theorem 8. *Let S be a subset of a topological space R and μ be a strictly positive*[3] *regular*[4] *measure of R. If there exists a normal open covering $\{U_\alpha\}$ of R such that each $U_\alpha \frown S$ is μ-measurable, then S is μ-measurable.*

Corollary 3. *Let f be a real-valued function of a topological space R and μ be a strictly positive regular measure. If there exists a normal open covering $\{U_\alpha\}$ of R such that each $f \mid U_\alpha$ is $(\mu \mid U_\alpha)$-measurable, then f is μ-measurable.*

References

[1] E. Michael: Local properties of topological spaces, Duke Math. J., **21**, 163–172 (1954).

[2] D. Montgomery: Non-separable spaces, Fund. Math., **25**, 527–533 (1935).

[3] K. Nagami: Baire sets, Borel sets and some typical semi-continuous functions, Nagoya Math. J., **7**, 85–93 (1954).

[4] A. H. Stone: Paracompactness and product spaces, Bull. Amer. Math. Soc., **54**, 977–982 (1948).

[5] J. W. Tukey: Convergence and uniformity in topology, Princeton (1940).

---

3) A measure μ of a topological space R is called strictly positive if for every open set G of R it holds that $\mu(G)>0$.

4) A measure μ of a topological space R is called regular if for every $A \subset R$ and for every $\varepsilon>0$ there exists an open set $G \supset A$ such that $\mu(G)-\mu(A)<\varepsilon$.