91. A Theorem on Paracompact Spaces

By Kiyoshi ISÉKI and Yasue MIYANAGA

(Comm. by K. KUNUGI, M.J.A., June 12, 1956)

Recently, K. Fujiwara [3] and K. Iséki [4] have shown that some properties on compact spaces are generalized very naturally to uniform spaces having Lebesgue property. In this paper, we shall extend a theorem of I. Gelfand and G. Silov [5] to paracompact space.

Let M be a metric space with metric ρ, and let $f(x)$ be a function defined on M. For a point $x \in M$, we shall define the *oscillation* of the function $f(x)$ at the point x. By $\omega(x, \varepsilon)$, we denote the least upper bound of $\rho(f(p), f(q))$ for $p, q\, S(x, \varepsilon)$, where $S(x, \varepsilon)$ is the sphere with center at x and radius ε. Then $\lim_{\varepsilon\to 0} \omega(x, \varepsilon)\ (=\omega(x))$ exists and this limit is called the *oscillation of the function* $f(x)$ *at the point* x.

It is well known that *a function* $f(x)$ *defined on a metric space is continuous at a point* x, *if and only if the oscillation of* $f(x)$ *at* x *is equal to zero* (see W. Sierpiński [6], p. 184).

I. Gelfand and G. Silov [5] proved the following proposition. *Let* $\varphi(x)$ *be a function defined on a compact set* M *in n-dimensional Euclidean space* R^n, *and let* $\omega(x)\leq\varepsilon$ *for every point* $x\in M$, *then, there is a continuous function* $f(x)$ *on* M *such that* $|f(x)-\varphi(x)|\leq 2\varepsilon$.

We shall extend the proposition by I. Gelfand and G. Silov to more general topological space. First of all, suppose that M is a compact metric space. By the compactness of M, we can find a positive number η such that $\rho(x', x'')<\eta$ implies $|\varphi(x')-\varphi(x'')|\leq 2\varepsilon$. The open covering $\{S(x, \eta)\mid x\in M\}$ of M has a finite covering $S(x_1, \eta), \cdots, S(x_n, \eta)$. Since M is a normal space, for the finite number of the open sets $S(x_i, \eta)$ $(i=1, 2, \cdots, n)$, there is such a decomposition $\lambda_i(x)$ $(i=1, 2, \cdots, n)$ of unity that

(1) each $\lambda_i(x)$ is a non-negative, continuous function on M,

(2) $1=\sum_{i=1}^{n}\lambda_i(x)$ for every x of M,

(3) $\lambda_i(x)=0$ on $M-S(x_i, \eta)$ $(i=1, 2, \cdots, n)$.

(See N. Bourbaki [2], p. 66.) To define a continuous function $f(x)$, let $f(x_i)=\varphi(x_i)$ and

$$f(x)=\sum_{i=1}^{n}\lambda_i(x)f(x_i),$$

then $f(x)$ is continuous on M. For any x of M, there is an open sphere $S(x_i, \eta)$ containing x_i.

$$|\varphi(x)-f(x)|=|\varphi(x)-\sum\lambda_i(x)f(x_i)|$$
$$=|\sum\lambda_i(x)(\varphi(x)-f(x_i))|\leq\sum\lambda_i(x)|\varphi(x)-f(x_i)|.$$

Since $\rho(x, x_j)\geq\eta$ implies $\lambda_j(x)=0$ and $\rho(x, x_i)<\eta$ implies $|\varphi(x)-f(x_i)|=|\varphi(x)-\varphi(x_i)|\leq 2\varepsilon$, we have $|\varphi(x)-f(x)|\leq\sum\lambda_i(x)|\varphi(x)-f(x_i)|\leq 2\varepsilon$. Therefore we have the following

Theorem 1. Let $\varphi(x)$ be a function defined on a compact metric space M and $\omega(x)\leq\varepsilon$ for every x of M, then, we can find a continuous function $f(x)$ such that

$$|\varphi(x)-f(x)|\leq 2\varepsilon.$$

Now, let $f(x)$ be a function defined on a topological space M. By the oscillation of a function $f(x)$ at a point x, we shall mean the number $\omega(x)=\inf_V \delta(f(V))$, where V runs over all neighbourhoods of x and $\delta(f(V))$ is the diameter of $f(V)$.

To extend Theorem 1, we shall consider a paracompact Hausdorff space M. Let $\varphi(x)$ be a function with the oscillation $\omega(x)\leq\alpha$ for every point x of M. Let ε be a given positive number. For each x of M, there is a neighbourhood $V(x)$ of x such that $\delta(f(V(x)))\leq\alpha+\varepsilon$. Then $\{V(x)\}_{x\in M}$ is a covering of M. By the paracompactness of M, the covering $\{U(x)\}_{x\in M}$ has locally finite refinement $\{U_\alpha\}$. As well known, for the covering $\{U_\alpha\}$, there is a decomposition of unity:

(4) There are non-negative continuous functions $f_\alpha(x)$ for each α.

(5) $f_\alpha(x)=0$ on $M-U_\alpha$.

(6) $\sum_\alpha f_\alpha(x)=1$ for every x of M.

(See, cf. R. G. Bartle and L. M. Graves [1], p. 401.)

Let x_α be a point of U_α, and let

$$f(x)=\sum_\alpha f_\alpha(x)\varphi(x_\alpha),$$

then $f(x)$ is continuous on M. For a point x of M, we have

$$|\varphi(x)-f(x)|=|\varphi(x)-\sum_\alpha f_\alpha(x)\varphi(x_\alpha)|$$
$$=|\sum f_\alpha(x)\ (\varphi(x)-\varphi(x_\alpha))|\leq\sum_\alpha f(x)|\varphi(x)-\varphi(x_\alpha)|.$$

Suppose that $x\in U_{\alpha_i}$ $(i=1, 2,\cdots, k)$, then

$$|\varphi(x)-\varphi(x_{\alpha_i})|\leq\alpha+\varepsilon$$

and, by (5), $f_\beta(y)=0$ $(\alpha_i\neq\beta)$, therefore

$$|\varphi(x)-f(x)|\leq\alpha+\varepsilon.$$

Hence we have the following

Theorem 2. Let M be a paracompact Hausdorff space, and $\varphi(x)$ a function on M with $\omega(x)\leq\alpha$. For a given positive number ε, there is a continuous function $f(x)$ such that

$$|f(x)-\varphi(x)|\leq\alpha+\varepsilon.$$

Some of the results of similar type for vector space valued functions have been obtained by S. Kasahara. The detail considerations will be appeared in his later paper.

References

[1] R. G. Bartle and L. M. Graves: Mappings between function spaces, Trans. Amer. Math. Soc., **72**, 400–413 (1952).
[2] N. Bourbaki: Topologie Générale, Chap. IX, Hermann, Paris.
[3] K. Fujiwara: Une note sur l'espace produit d'un espace topologique par luimême, Math. Jour. Okayama Univ., **5**, 121–125 (1956).
[4] K. Iséki: Notes on topological spaces. II. Some properties of topological spaces with Lebesgue property, Proc. Japan Acad., **32**, 171–173 (1956).
[5] I. Gelfand and G. Silov: Über verschiedene Methoden der Einführung der Topologie in die Menge der maximalen Ideale eines normierten Ringes, Recueil Math., **51**, 59–73 (1941).
[6] W. Sierpiński: General Topology (2nd Ed.), Toronto (1952).