88. *Note on Algebras of Strongly Unbounded Representation Type*[1]

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§1. Let A be an associative algebra with a unit element over an algebraically closed field k and $g_A(d)$ be the number of inequivalent indecomposable representations of A of degree d where d is a positive integer. Then if A has indecomposable representations of arbitrary high degrees and $g_A(d)=\infty$ for an infinite number of integers d, A is said to be of *strongly unbounded representation type.* In a paper [1] James P. Jans proved that the following four conditions are sufficient for an algebra to be of strongly unbounded representation type:

(*1*) *L_A, the two-sided ideal lattice, is infinite.*

(*2*) *For any i and any two-sided ideal A_0 in N (N is the radical of A) e_iA_0 (A_0e_i) has more than three covers in e_iN (Ne_i) where A' is said to be the cover of e_iA if $A'\supset e_iA_0$ and $A'\supset B\supseteqq e_iA_0$ implies $B=e_iA_0$.*

(*3*) *The graph $G(A_0)$ associated with any two-sided ideal $A_0\subset N$ is a cycle where the graph $G(A_0)$ is such a set $\{P_1, P_1 \& P_2, P_2, P_2 \& P_3, P_3, \cdots, P_{n-1}, P_{n-1} \& P_n, P_n\}$[2] that $P_i \& P_j$ holds if $e_iA'e_j$ covers $e_iA_0e_j$ for some cover A' of A_0 and $G(A_0)$ is said to be the cycle if $\{G(A_0), G(A_0)\}$ is also a graph.*

(*4*) *The graph $G(A_0)$ associated with any two-sided ideal $A_0\subset N$ branches at each end where $G_1(A_0)$ is said to extend $G_2(A_0)$ at the right end if $\{G_2(A_0), G_1(A_0)\}$ is the graph and $G(A_0)$ is said to branch at one end if it is extended by at least two distinct graphs at one end.*

Now in this paper we shall prove that the following two conditions are also sufficient for an algebra to be of strongly unbounded representation type:

(*5*) *The graph $G(A_0)$ associated with any two-sided ideal $A_0\subset N$ is*

$$\left\{\begin{matrix} P_{r_2}, P_{k_1} \,\&\, P_{r_2}, P_{k_1}, P_{k_1} \,\&\, P_{r_1}, \\ P_{r_3}, P_{k_2} \,\&\, P_{r_3}, P_{k_2}, P_{k_2} \,\&\, P_{r_1}, \end{matrix} P_{r_1}, P_{k_3} \,\&\, P_{r_1}, P_{k_3}, P_{k_3} \,\&\, P_{r_4}, P_{r_4}\right\}.$$

(*6*) *The graph $G(A_0)$ is*

$$\left\{P_{k_5}, P_{k_5} \,\&\, P_{j_4}, P_{j_4}, P_{k_4} \,\&\, P_{j_4}, P_{k_4}, P_{k_4} \,\&\, P_{j_3}, P_{j_3}, P_{k_3} \,\&\, P_{j_3}, \begin{matrix} P_{k_3}, P_{k_3} \,\&\, P_{j_2}, \\ P_{k_2}, P_{k_2} \,\&\, P_{j_2}, \end{matrix} P_{j_2}, P_{k_1} \,\&\, P_{j_2}, P_{k_1}, P_{k_1} \,\&\, P_{j_1}, P_{j_1}\right\}.$$

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1) James P. Jans [1].

2) $P_1, P_2, \cdots, P_n$ mean vertices, and "P_i & P_j" means that "P_i and P_j are connected by an (oriented) edge".

§2. In this chapter we shall prove the following

Theorem 1. *If the graph $G(A_0)$ associated with $A_0 \subset N$ is* $\left\{\begin{matrix} P_{r_2}, P_{k_1} \,\&\, P_{r_2}, P_{k_1}, P_{k_1} \,\&\, P_{r_1}, \\ P_{r_3}, P_{k_2} \,\&\, P_{r_3}, P_{k_2}, P_{k_2} \,\&\, P_{r_1}, \end{matrix} P_{r_1}, P_{k_3} \,\&\, P_{r_1}, P_{k_3}, P_{k_3} \,\&\, P_{r_4}, P_{r_4} \right\}$, *then A is of strongly unbounded representation type.*

Proof. In order to simplify the proof we assume that A is the basic algebra and $N^2=0$. In the case where $N^2 \neq 0$, we can prove by the same way as [1]. First we have the representation

$$R_{ij}(a)=\begin{Bmatrix} X_{k_j}(a) & \\ Y_{ij}(a) & X_{r_t}(a) \end{Bmatrix}$$

for P_{k_t} & P_{r_j} where $R_{ij}(a_{ij})$ has only 1 in the lower corner, namely $Y_{ij}(a_{ij})=1$ and $X_{s_t}(a_{ij})=0$, for $a_{ij} \in e_i N e_j$ and $R_{ij}(a_{st})=0$ if $(i, j) \neq (s,t)$.

Next we construct the matrix function R_{cs}, $c \in k$ and S is an integer,

$$R_{cs}(a)=\begin{Bmatrix} X_T(a) & \\ Y(a) & X_B(a) \end{Bmatrix},$$

as follows.

Let $X_T(a)$ be the direct sum of $I_{2s} * X_{k_i}(a)$ $(i=1, 2, 3)$ and let $X_B(a)$ be the direct sum of $I_{3s} * X_{r_1}(a)$ and $I_s * X_{r_i}(a)$ $(i=2, 3, 4)$. Then it is clear that $X_T(a)$ and $X_B(a)$ are representations of A.

Next we put

$$M_1=\begin{bmatrix} 0\ 0\cdots0 & 0\cdots\cdots0 \\ \cdots & \cdots \\ 0\cdots\cdots0 & 0\cdots\cdots0 \\ 1 \qquad & 0\cdots\cdots0 \\ \ \ 1 & \cdots \\ \quad \ddots & \\ \qquad 1 & 0\cdots\cdots0 \\ 0\cdots\cdots0 & 1 \qquad \\ \cdots & \ \ 1 \\ & \quad \ddots \\ \underbrace{0\cdots\cdots0}_{S} & \underbrace{\qquad 1}_{S} \end{bmatrix}, \qquad M_2=\begin{bmatrix} 1 \qquad & 0\cdots\cdots0 \\ \ \ 1 & \cdots \\ \quad \ddots & \\ \qquad 1 & 0\cdots\cdots0 \\ 0\cdots\cdots0 & 1 \qquad \\ \cdots & \ \ 1 \\ & \quad \ddots \\ 0\cdots\cdots0 & \qquad 1 \\ 0\cdots\cdots0 & 0\cdots\cdots0 \\ \cdots & \cdots \\ 0\cdots\cdots0 & 0\cdots\cdots0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\}S \\ \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\}S, \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\}S \end{matrix}$$

$$M_3=\begin{bmatrix} 1 \qquad & 0\cdots\cdots0 \\ \ \ 1 & \cdots \\ \quad \ddots & \\ \qquad 1 & 0\cdots\cdots0 \\ 0\cdots\cdots0 & 0\cdots\cdots0 \\ \cdots & \cdots \\ 0\cdots\cdots0 & 0\cdots\cdots0 \\ 0\cdots\cdots0 & 1 \qquad \\ \cdots & \ \ 1 \\ & \quad \ddots \\ \underbrace{0\cdots\cdots0}_{S} & \underbrace{\qquad 1}_{S} \end{bmatrix}, \qquad \begin{matrix} M_4=\begin{bmatrix} \overbrace{\begin{matrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{matrix}}^{S} & \overbrace{\begin{matrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{matrix}}^{S} \end{bmatrix} \\ M_{5c}=\begin{bmatrix} \underbrace{\begin{matrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{matrix}}_{S} & \underbrace{\begin{matrix} c & & & & & & \\ 1 & c & & & & & \\ & 1 & c & & & & \\ & & 1 & \ddots & & & \\ & & & 1 & c & & \\ & & & & 1 & c & \\ & & & & & 1 & c \end{matrix}}_{S} \end{bmatrix} \end{matrix}$$

Let $Y(a)$ have $M_1{}^*Y_{11}(a)$ directly below $I_{2s}{}^*X_{k_1}(a)$ and directly to the left of $I_{3s}{}^*X_{r_1}(a)$, $M_2{}^*Y_{12}(a)$ directly below $I_{2s}{}^*X_{k_2}(a)$ and directly to the left of $I_{3s}{}^*X_{r_1}(a)$, $M_3{}^*Y_{13}(a)$ directly below $I_{2s}{}^*X_{k_3}(a)$ and directly to the left of $I_{3s}{}^*X_{r_1}(a)$, $M_4{}^*Y_{21}(a)$ directly below $I_{2s}{}^*X_{k_1}(a)$ and directly to the left[3] of $I_s{}^*X_{r_2}(a)$, $M_4{}^*Y_{32}(a)$ directly below $I_{2s}{}^*X_{k_2}(a)$ and directly to the left of $I_s{}^*X_{r_3}(a)$ and $M_5{}^*Y_{43}(a)$ directly below $I_{2s}{}^*X_{k_3}(a)$ and directly to the left of $I_s{}^*X_{r_4}(a)$. Fill out the rest with zeroes.

Now it is shown by the computation of eigenvalues of any commutator of $R_{cs}(a)$ that $R_{cs}(a)$ is a directly indecomposable representation.

Next we shall prove that A is of strongly unbounded representation type. For this purpose we have only to prove that $R_{cs}(a)$ and $R_{ds}(a)$ can not be similar for $c \neq d$. Now suppose they were similar. Then there would exist a non-singular matrix P intertwining R_{cs} and R_{ds} and we divide P into submatrices,

$$P=\begin{bmatrix} P_{11} & P_{12} & \cdots\cdots & P_{17} \\ P_{21} & P_{22} & \cdots\cdots & P_{27} \\ & \cdots\cdots\cdots\cdots & & \\ P_{71} & P_{72} & \cdots\cdots & P_{77} \end{bmatrix},$$

corresponding to the divisions of R_{cs}. It is clear from $R_{cs}(a)P=PR_{ds}(a)$ that $P_{1i}=0$ $(i \neq 1)$, $P_{2i}=0$ $(i \neq 2)$, $P_{3i}=0$ $(i \neq 3)$ and P_{45}, P_{46}, P_{47}, P_{54}, P_{56}, P_{57}, P_{64}, P_{65}, P_{67}, P_{74}, P_{75} and P_{76} are zero, and $M_1P_{11}=P_{44}M_1$, $M_2P_{22}=P_{44}M_2$, $M_3P_{33}=P_{44}M_3$, $M_4P_{11}=P_{55}M_4$, $M_4P_{22}=P_{66}M_4$ and $M_{5c}P_{33}=P_{77}M_{5d}$. Hence P_{11}, P_{22}, P_{33} and P_{44} are the direct sums of P_{77} and $P_{55}=P_{66}=P_{77}$ and from $M_{5c}P_{33}=P_{77}M_{5d}$, P_{77} can not be non-singular if $c \neq d$.[4]

§3. In this chapter we shall prove the following

Theorem 2. *If the graph $G(A_0)$ associated with $A_0 \subset N$ is* $\left\{P_{k_5}, P_{k_5} \,\&\, P_{j_4}, P_{j_4}, P_{k_4} \,\&\, P_{j_4}, P_{k_4}, P_{k_4} \,\&\, P_{j_3}, P_{j_3}, P_{k_3} \,\&\, P_{j_3}, P_{k_3}, P_{k_3} \,\&\, P_{k_2}, P_{k_2} \,\&\, P_{j_2}, P_{j_2}, P_{j_2}, P_{k_1} \,\&\, P_{j_2}, P_{k_1}, P_{k_1} \,\&\, P_{j_1}, P_{j_1}\right\}$, *then A is of strongly unbounded representation type.*

Proof. By the same way as Theorem 1 we construct the matrix function R_{cs}, as follows,

$$R_{cs}(a)=\begin{Bmatrix} X_T(a) & \\ Y(a) & X_B(a) \end{Bmatrix}.$$

Let $X_T(a)$ be the direct sum of $I_{2s}{}^*X_{j_1}(a)$, $I_{6s}{}^*X_{j_2}(a)$, $I_{4s}{}^*X_{j_3}(a)$ and $I_{2s}{}^*X_{j_4}(a)$ and let $X_B(a)$ be the direct sum of $I_{4s}{}^*X_{k_1}(a)$, $I_{3s}{}^*X_{k_2}(a)$, $I_{5s}{}^*X_{k_3}(a)$, $I_{3s}{}^*X_{k_4}(a)$ and $I_s{}^*X_{k_5}(a)$. Then $X_T(a)$ and $X_B(a)$ are all representations of A.

Next we put

3) $*$ means the Kronecker product.

4) The computation is long and we shall omit it. Cf. T. Yoshii [2].

$$M_{1c}=\begin{bmatrix}
1 & c & & & & & \\
1 & & & & & & \\
1 & 1 & & & & & \\
 & 1 & & & & & \\
 & 1 & 1 & c & & & \\
 & & 1 & & & & \\
 & & 1 & 1 & & & \\
 & & & 1 & & & \\
 & & & 1 & \ddots & & \\
 & & & & & 1 & c \\
 & & & & & 1 & \\
 & & & & & 1 & 1 \\
 & & & & & & 1
\end{bmatrix},\qquad
M_2=\begin{array}{c}
\begin{array}{cc}\overbrace{\qquad\qquad\qquad\qquad}^{3s} & \overbrace{\qquad\qquad\qquad\qquad}^{3s}\end{array}\\
\begin{bmatrix}
100 & & & & 000 & & & \\
010 & & & & 000 & & & \\
000 & & & & 010 & & & \\
000 & & & & 001 & & & \\
 & 100 & & & & 000 & & \\
 & 010 & & & & 000 & & \\
 & 000 & & & & 010 & & \\
 & 000 & \ddots & & & 001 & \ddots & \\
 & & & 100 & & & & 000 \\
 & & & 010 & & & & 000 \\
 & & & 000 & & & & 010 \\
 & & & 000 & & & & 001
\end{bmatrix}
\end{array},$$

$$M_3=\begin{bmatrix}
0\ 0\ 0 & & & & 1\ 0\ 0 & & & \\
0\ 0\ 0 & & & & 0\ 1\ 0 & & & \\
0\ 0\ 1 & & & & 0\ 0\ 0 & & & \\
 & 0\ 0\ 0 & & & & 1\ 0\ 0 & & \\
 & 0\ 0\ 0 & & & & 0\ 1\ 0 & & \\
 & 0\ 0\ 1 & \ddots & & & 0\ 0\ 0 & \ddots & \\
 & & & 0\ 0\ 0 & & & & 1\ 0\ 0 \\
 & & & 0\ 0\ 0 & & & & 0\ 1\ 0 \\
 & & & 0\ 0\ 1 & & & & 0\ 0\ 0
\end{bmatrix},$$

$$M_4=\begin{bmatrix}
1\ 0\ 0 & & & 0\ 0\ 0 & & \\
0\ 0\ 0 & & & 1\ 0\ 0 & & \\
0\ 1\ 0 & & & 0\ 0\ 0 & & \\
0\ 0\ 1 & & & 0\ 0\ 0 & & \\
0\ 0\ 0 & \ddots & & 0\ 0\ 1 & \ddots & \\
 & & 1\ 0\ 0 & & & 0\ 0\ 0 \\
 & & 0\ 0\ 0 & & & 1\ 0\ 0 \\
 & & 0\ 1\ 0 & & & 0\ 0\ 0 \\
 & & 0\ 0\ 1 & & & 0\ 0\ 0 \\
 & & 0\ 0\ 0 & & & 0\ 0\ 1
\end{bmatrix},$$

$$M_5=\begin{array}{c}
\begin{array}{cc}\overbrace{\qquad\qquad}^{2s} & \overbrace{\qquad\qquad}^{2s}\end{array}\\
\begin{bmatrix}
1\ 0 & & & 0\ 0 & & \\
0\ 0 & & & 1\ 0 & & \\
0\ 1 & & & 0\ 0 & & \\
0\ 0 & & & 0\ 1 & & \\
0\ 0 & \ddots & & 0\ 1 & \ddots & \\
 & & 1\ 0 & & & 0\ 0 \\
 & & 0\ 0 & & & 1\ 0 \\
 & & 0\ 1 & & & 0\ 0 \\
 & & 0\ 0 & & & 0\ 1 \\
 & & 0\ 0 & & & 0\ 1
\end{bmatrix}
\end{array},\qquad
M_6=\begin{bmatrix}
1\ 0 & & & 0\ 0 & & \\
0\ 0 & & & 1\ 0 & & \\
0\ 1 & \ddots & & 0\ 0 & \ddots & \\
 & & 1\ 0 & & & 0\ 0 \\
 & & 0\ 0 & & & 1\ 0 \\
 & & 0\ 1 & & & 0\ 0
\end{bmatrix},$$

$$M_7=\begin{array}{c}\overbrace{\qquad}^{s}\ \overbrace{\qquad}^{s}\\ \left[\begin{array}{cccc}1 & & 0 & \\ 0 & & 1 & \\ 0 & & 1 & \\ & \ddots & & \ddots \\ & 1 & & 0 \\ & 0 & & 1 \\ & 0 & & 1\end{array}\right]\end{array} \quad\text{and}\quad M_8=\left[\begin{array}{ccccc}1 & & & & 0\cdots\cdots 0\\ & 1 & & & \vdots \qquad \vdots\\ & & 1 & & \vdots \qquad \vdots\\ & & & \ddots & \vdots \qquad \vdots\\ & & & 1 & 0\cdots\cdots 0\end{array}\right].$$

Then let $Y(a)$ have $M_1{}^*Y_{11}$ directly below $I_{2s}{}^*X_{j_1}(a)$ and directly to the left of $I_{4s}{}^*X_{k_1}(a)$, $M_2{}^*Y_{12}(a)$ directly below $I_{6s}{}^*X_{j_2}(a)$ and directly to the left of $I_{4s}{}^*X_{k_1}(a)$, $M_3{}^*Y_{22}(a)$ directly below $I_{6s}{}^*X_{j_2}(a)$ and directly to the left of $I_{3s}{}^*X_{k_2}(a)$, $M_4{}^*Y_{32}(a)$ directly below $I_{6s}{}^*X_{j_2}(a)$ and directly to the left of $I_{5s}{}^*X_{k_3}(a)$, $M_5{}^*Y_{33}(a)$ directly below $I_{4s}{}^*X_{j_3}(a)$ and directly to the left of $I_{5s}{}^*X_{k_3}(a)$, $M_6{}^*Y_{43}(a)$ directly below $I_{4s}{}^*X_{j_3}(a)$ and directly to the left of $I_{3s}{}^*X_{k_4}(a)$, $M_7{}^*Y_{44}(a)$ directly below $I_{2s}{}^*X_{j_4}(a)$ and directly to the left of $I_{3s}{}^*X_{k_4}(a)$ and $M_8{}^*Y_{54}(a)$ directly below $I_{2s}{}^*X_{j_4}(a)$ and directly to the left of $I_s{}^*X_{k_5}(a)$.

Now it is shown by the same way as above that R_{cs} is a directly indecomposable representation of A.

Next in order to show that A is of strongly unbounded representation type, we prove that R_{cs} and R_{ds} can not be similar for $c\neq d$. Now suppose they were similar. Then there would exist a non-singular matrix P intertwining R_{cs} and R_{ds}. Let P be divided into submatrices,

$$P=\left[\begin{array}{cccc}P_{11} & P_{12} & P_{13}\cdots P_{19}\\ P_{21} & P_{22} & P_{23}\cdots P_{29}\\ & \cdots\cdots\cdots & \\ P_{91} & P_{82} & P_{93}\cdots P_{99}\end{array}\right],$$

corresponding to the divisions of R_{cs}. It is clear that P_{1i} $(i\neq 1)$, P_{2i} $(i\neq 2)$, P_{3i} $(i\neq 3)$, P_{4i} $(i\neq 4)$, P_{56}, P_{57}, P_{58}, P_{59}, P_{65}, P_{67}, P_{68}, P_{69}, P_{75}, P_{76}, P_{78}, P_{79}, P_{85}, P_{86}, P_{87}, P_{89}, P_{95}, P_{93}, P_{97} and P_{98} are zero. Moreover we have

$$P_{55}=\left[\begin{array}{ccccccccccc}a_{11} & & & & a_{15} & & & & a_{19} & & \\ & a_{22} & & & & a_{26} & & & & \cdot & \\ & & a_{33} & & & & a_{37} & & & & \cdot \\ & & & a_{44} & & & & a_{48} & & & \cdot \\ a_{51} & & & & a_{55} & & & & a_{59} & & \\ & a_{62} & & & & a_{66} & & & & \cdot & \\ & & a_{73} & & & & a_{77} & & & & \cdot \\ & & & \cdot & & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & & & & \cdot & & \end{array}\right]$$

and it is impossible from $PR_{cs}(a_{11})=R_{ds}(a_{11})P$ that P is non-singular. Thus this proof is completed.

References

[1] James P. Jans: On the Indecomposable Representation of Algebras, Dissertation, University of Michigan (1954).

[2] T. Yoshii: On algebras of bounded representation type, Osaka Math. Jour., **8**, No. 1 (1956) (forthcoming).