87. *Some Strong Summability of Fourier Series. II*

By Masakiti KINUKAWA

Natural Science Division, International Christian University, Tokyo

(Comm. by Z. SUETUNA, M.J.A., June 12, 1956)

1. Let $u(x)$ be integrable L^p $(p>1)$, periodic with period 2π and let $s_n(x)$ be the nth partial sum of its Fourier series. Then S. Izumi [5] proved that *if* $p\geqq k>1$, $\varepsilon>0$ *and*

$$\left(\int_{-\pi}^{\pi}|u(x+t)-u(x)|^p dx\right)^{1/p}\leqq K\{t^{1/k}(\log 1/t)^{-(1+\varepsilon)/k}\},$$

then the series

$$\sum_{n=1}^{\infty}|s_n(x)-u(x)|^k \tag{1.1}$$

converges for almost all x. Concerning the convergence of the series (1.1), S. Izumi [4] and the author [6, 7] have gotten some related results.

In this paper, we shall prove more general theorems concerning the series (1.1), replacing the partial sum $s_n(x)$ by the Cesàro mean $\sigma_n^\delta(x)$.

2. Suppose that

$$f(z)=\sum_{n=0}^{\infty}c_n z^n,\quad (z=re^{i\theta}),$$

is an analytic function of z, regular for $|z|=r<1$ and its boundary function is $f(e^{i\theta})$. Then we say that[1] $f(z)$ belongs to the "complex" class Lip (α,β,p) if it satisfies

$$M_p(r,f')=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f'(re^{i\theta})|^p d\theta\right)^{1/p}=O\left\{(1-r)^{-1+\alpha}\left(\log\frac{1}{1-r}\right)^{-\beta}\right\}.$$

Throughout this paper we use the following notation:

$$\sigma_n^0(\theta)=s_n(\theta)=\sum_{\nu=0}^{n}c_\nu e^{i\nu\theta},$$

$$\sigma_n^\delta(\theta)=\frac{1}{A_n^\delta}\sum_{\nu=0}^{n}A_{n-\nu}^{\delta-1}s_\nu(\theta),\quad \text{for } \delta>-1,$$

$$t_n(\theta)=nc_n e^{in\theta},$$

$$\tau_n^\delta(\theta)=\frac{1}{A_n^\delta}\sum_{\nu=0}^{n}A_{n-\nu}^{\delta-1}t_\nu(\theta),\quad \delta>0,$$

where

$$A_n^\delta=\binom{n+\delta}{n}\sim\frac{n^\delta}{\Gamma(\delta+1)}.$$

Then we have $\tau_n^\delta(\theta)=n\{\sigma_n^\delta(\theta)-\sigma_{n-1}^\delta(\theta)\}=\delta\{\sigma_n^{\delta-1}(\theta)-\sigma_n^\delta(\theta)\}$.

Our results may now be stated as follows:

---

1) Cf. Hardy-Littlewood [2].

Theorem 1. *If $f(z)$ belongs to the class Lip (α, β, p), then the series*

$$\sum_{n=1}^{\infty} |\sigma_n^{\delta}(\theta)-f(e^{i\theta})|^k$$

converges for almost all θ, where $2\geqq p\geqq k>1$, $\alpha=1/k$ and $\beta>1/k$ or $\beta>1/p+1/k$ according as $\delta>1/p-1$ or $\delta=1/p-1$.

Theorem 2. *If $f(z)$ belongs to the class Lip (α, β, p), then the series*

$$\sum_{n=1}^{\infty} |\tau_n^{\delta}(\theta)|^k$$

converges for almost all θ, where $2\geqq p\geqq k>1$, $\alpha=1/k$ and $\beta>1/k$ or $\beta>1/k+1/p$ according as $\delta>1/p$ or $\delta=1/p$.

Theorem 3. *If $f(z)$ belongs to the class Lip (α, p), i.e. Lip $(\alpha, 0, p)$, then the series*

$$\sum_{n=2}^{\infty} \frac{|\sigma_n^{\delta}(\theta)-f(e^{i\theta})|^k}{n^a(\log n)^b}$$

converges for almost all θ, where $2\geqq p>1$, $p\geqq k>0$, $1\geqq \alpha>0$, $a=1-k\alpha$ and $b>1$ or $b>1+k/p$ according as $\delta>1/p-1$ or $\delta=1/p-1$.

Theorem 4. *If $f(z)$ belongs to the class Lip (α, p), then the series*

$$\sum_{n=2}^{\infty} |\tau_n^{\delta}(\theta)|^k n^{-a}(\log n)^{-b}$$

converges for almost all θ, where $2\geqq p>1$, $p\geqq k>0$, $1\geqq \alpha>0$, $a=1-k\alpha$, and $b>1$ or $b>1+k/p$ according as $\delta>1/p$ or $\delta=1/p$.

3. Let $u(\theta)$ be a real integrable function, periodic with period 2π, and let its Fourier series be

$$a_0/2+\sum_{n=1}^{\infty}(a_n \cos n\theta+b_n \sin n\theta).$$

If we put $c_0=a_0/2$, $c_n=a_n-ib_n$ $(n>0)$, then

$$f(z)=\sum_{n=0}^{\infty} c_n z^n$$

is regular for $|z|=r<1$. It can be proved that[2)] if $u(\theta)$ belongs to the "real" class Lip (α, β, p), that is,

$$\left(\int_{-\pi}^{\pi}|u(\theta+t)-u(\theta)|^p d\theta\right)^{1/p}=O\{t^{\alpha}(\log 1/t)^{-\beta}\},$$

where $0<\alpha<1$, $\beta\geqq 0$ and $p>1$, then $f(z)$ belongs also to the complex class Lip (α, β, p). So that we can easily deduce from above theorems some analogous theorems for the real function $u(\theta)$. Hence we need not state those here.

4. Before preceding to prove Theorem 1, it is convenient to state a lemma.

Lemma 1. *If $f(z)$ belongs to the class Lip (α, β, p), then*

$$(4.1)\qquad \left(\int_{-\pi}^{\pi}|f(re^{i\theta})-f(e^{i\theta})|^p d\theta\right)^{1/p}=O\left\{(1-r)^{\alpha}\left(\log\frac{1}{1-r}\right)^{-\beta}\right\},$$

2) Cf. Hardy-Littlewood [2, 3] and Loo [8].

$$(4.2)\quad \left(\int_{-\pi}^{\pi}|f(re^{i(\theta+t)})-f(re^{i\theta})|^p\,d\theta\right)^{1/p}=O\left\{t(1-r)^{-1+\alpha}\left(\log\frac{1}{1-r}\right)^{-\beta}\right\},$$

where $1\geqq\alpha>0$, $\beta\geqq 0$ *and* $p>1$.

For, the left hand side of (4.1) is less than

$$\left\{\int_{-\pi}^{\pi}\left(\int_r^1|f'(\rho e^{i\theta})|\,d\rho\right)^p d\theta\right\}^{1/p},$$

which is dominated by the following, using the Minkowski inequality,

$$\int_r^1 d\rho\left(\int_{-\pi}^{\pi}|f'(\rho e^{i\theta})|^p\,d\theta\right)^{1/p}=O\left\{\int_r^1(1-\rho)^{-1+\alpha}\left(\log\frac{1}{1-\rho}\right)^{-\beta}d\rho\right\}$$

$$=O\left\{(1-r)^{\alpha}\left(\log\frac{1}{1-r}\right)^{-\beta}\right\}.$$

Thus we get (4.1). The left hand side of (4.2) is dominated by[3]

$$\left\{\int_{-\pi}^{\pi}\left(\int_{\theta}^{t+\theta}|f'(re^{ix})|\,dx\right)^p d\theta\right\}^{1/p}$$

$$\leqq\left\{\int_{-\pi}^{\pi}\left[\left(\int_{\theta}^{\theta+t}|f'(re^{ix})|^p\,dx\right)^{1/p}\left(\int_{\theta}^{\theta+t}dx\right)^{1/p'}\right]^p d\theta\right\}^{1/p}$$

$$\leqq K\,t^{1/p'}\left\{\int_{-\pi}^{\pi}\int_{\theta}^{\theta+t}|f'(re^{ix})|^p\,dx\,d\theta\right\}^{1/p}$$

$$\leqq K\,t^{1/p'+1/p}\left(\int_{-\pi}^{\pi}|f'(re^{ix})|^p\,dx\right)^{1/p}\leqq K\,t(1-r)^{-1+\alpha}\left(\log\frac{1}{1-r}\right)^{-\beta},$$

which is the right side of (4.2).

We are now in position to prove Theorem 1. We have

$$\sum_{n=0}^{\infty}A_n^{\delta}\{\sigma_n^{\delta}(\theta)-f(e^{i\theta})\}z^n=[f(ze^{i\theta})-f(e^{i\theta})]/(1-z)^{\delta+1},\quad (z=re^{it}),$$

$$\equiv G(t)H(t),\ \text{say}.$$

Let $h=1-r$, then by the Hausdorff-Young theorem,

$$\left\{\sum_{n=1}^{\infty}|A_n^{\delta}[\sigma_n^{\delta}(\theta)-f(e^{i\theta})]r^n\sin nh|^{p'}\right\}^{p/p'}$$

$$\leqq K\left\{\int_{-\pi}^{\pi}|G(t+h)H(t+h)-G(t-h)H(t-h)|^p\,dt\right\}$$

$$\leqq K\left\{\int_{-\pi}^{\pi}|H(t+h)|^p|G(t+h)-G(t-h)|^p dt+\int_{-\pi}^{\pi}|G(t-h)|^p\,|H(t+h)-H(t-h)|^p\,dt\right\}=K\,\{J_1(\theta)+J_2(\theta)\},\ \text{say}.$$

First we consider the case $\delta>1/p-1$. We have, by Lemma 1,

$$\int_{-\pi}^{\pi}J_1(\theta)d\theta\leqq K\int_{-\pi}^{\pi}[(1-r)^2+(t+h)^2]^{-p(\delta+1)/2}dt\int_{-\pi}^{\pi}|f(re^{i(\theta+t+h)})-f(re^{i(\theta+t-h)})|^p d\theta$$

$$\leqq K\int_{-\pi}^{\pi}[(1-r)^2+(t+h)^2]^{-p(\delta+1)/2}\left\{h(1-r)^{-1+\alpha}\left(\log\frac{1}{1-r}\right)^{-\beta}\right\}^p dt$$

3) We denote by K an absolute constant, which is not necessarily the same in different occurrences.

$$\leqq Kh^p(1-r)^{p(-1+\alpha)}\left(\log\frac{1}{1-r}\right)^{-p\beta}\left\{\int_{-ch}^{ch}(1-r)^{-p(\delta+1)}dt + \left[\int_{-\pi}^{-ch}+\int_{ch}^{\pi}\right](t+h)^{-p(\delta+1)}dt\right\}$$

$$\leqq K(1-r)^{1+p\alpha-p(\delta+1)}\left(\log\frac{1}{1-r}\right)^{-p\beta}, \tag{4.3}$$

since $1-p(\delta+1)<0$.

We have also

$$\int_{-\pi}^{\pi}J_2(\theta)d\theta\leqq\int_{-\pi}^{\pi}|(H(t+h)-H(t-h)|^p\,dt\int_{-\pi}^{\pi}|f(re^{i(\theta+t-h)})-f(e^{i\theta})|^p\,d\theta$$

$$\leqq\int_{-\pi}^{\pi}|H(t+h)-H(t-h)|^p\,dt\left\{\int_{-\pi}^{\pi}|f(re^{i(\theta+t-h)})-f(re^{i\theta})|^p + |f(re^{i\theta})-f(e^{i\theta})|^p\,d\theta\right\}$$

$$\leqq K\left[(1-r)^{-1+\alpha}\left(\log\frac{1}{1-r}\right)^{-\beta}\right]^p\int_{-\pi}^{\pi}|t-h|^p\,|H(t+h)-H(t-h)|^p\,dt + K\left[(1-r)^{\alpha}\left(\log\frac{1}{1-r}\right)^{-\beta}\right]^p\int_0^{\pi}[(1-r)^2+t^2]^{-p(\delta+1)/2}\,dt.$$

The second term of the right side is less than

$$K(1-r)^{1+p\alpha-p(\delta+1)}\left(\log\frac{1}{1-r}\right)^{-p\beta},$$

and the first term is

$$K\left[(1-r)^{-1+\alpha}\left(\log\frac{1}{1-r}\right)^{-\beta}\right]^p\int_{-\pi-h}^{\pi-h}|t|^p\,|H(t+2h)-H(t)|^p\,dt$$

$$\leqq K\left[(1-r)^{-1+\alpha}\left(\log\frac{1}{1-r}\right)^{-\beta}\right]^p\left\{\int_{-2ch}^{2ch}(1-r)^{-p(\delta+1)}t^p\,dt + \left[\int_{-\pi-h}^{-2ch}+\int_{2ch}^{\pi}\right]h^p\,|t|^{p-(\delta+2)p}\,dt\right\}$$

$$\leqq K\left[(1-r)^{-1+\alpha}\left(\log\frac{1}{1-r}\right)^{-\beta}\right]^p\left\{(1-r)^{-p(\delta+1)}h^{p+1}+h^{p+1-(\delta+1)p}\right\}$$

$$\leqq K(1-r)^{1+p\alpha-p(\delta+1)}\left(\log\frac{1}{1-r}\right)^{-p\beta}.$$

Thus we get

$$\int_{-\pi}^{\pi}J_2(\theta)\,d\theta\leqq K(1-r)^{1+p\alpha-p(\delta+1)}\left(\log\frac{1}{1-r}\right)^{-p\beta}. \tag{4.4}$$

Using the above estimations and taking $h=1-r=\pi/2^{\lambda+1}$, we get

$$\int_{-\pi}^{\pi}\left\{\sum_{n=2^{\lambda-1}+1}^{2^\lambda}n^{p'\delta}\,|\sigma_n^\delta(\theta)-f(e^{i\theta})|^{p'}\right\}^{p/p'}d\theta\leqq\int_{-\pi}^{\pi}[J_1(\theta)+J_2(\theta)]d\theta$$

$$\leqq K2^{\lambda p(\delta+p-p\alpha-1)}\lambda^{-p\beta}.$$

Hence

$$(4.5)\qquad \int_{-\pi}^{\pi}\Big\{\sum_{n=2^{\lambda-1}+1}^{2^\lambda} |\sigma_n^\delta(\theta)-f(e^{i\theta})|^{p'}\Big\}^{p/p'} d\theta \leqq K 2^{\lambda(p-p\alpha-1)}\lambda^{-p\beta}.$$

So that we have, by the Hölder inequality,

$$\sum_{n=2}^{\infty}\int_{-\pi}^{\pi}|\sigma_n^\delta(\theta)-f(e^{i\theta})|^k d\theta=\sum_{\lambda=1}^{\infty}\int_{-\pi}^{\pi}\Big\{\sum_{n=2^{\lambda-1}+1}^{2^\lambda}|\sigma_n^\delta(\theta)-f(e^{i\theta})|^k\Big\}d\theta$$

$$\leqq \sum_{\lambda=1}^{\infty}2^{\lambda/q'}\int_{-\pi}^{\pi}\Big\{\sum_{n=2^{\lambda-1}+1}^{2^\lambda}|\sigma_n^\delta(\theta)-f(e^{i\theta})|^{kq}\Big\}^{1/q}d\theta,$$

where $q=p'/k$ and $q'=q/(q-1)$. Then by (4.5), the last sum is less than

$$\sum_{\lambda=1}^{\infty}2^{\lambda/q'}\Big(\int_{-\pi}^{\pi}\Big\{\sum_{n=2^{\lambda-1}+1}^{2^\lambda}|\sigma_n^\delta(\theta)-f(e^{i\theta})|^{p'}\Big\}^{p/p'}d\theta\Big)^{k/p}$$

$$\leqq K\sum_{\lambda=1}^{\infty}2^{\lambda[1/q'+(p-p\alpha-1)k/p]}\lambda^{-k\beta}$$

$$=K\sum_{\lambda=1}^{\infty}2^{\lambda[1-k\alpha]}\lambda^{-k\beta}=K\sum_{\lambda=1}^{\infty}\lambda^{-k\beta}<\infty.$$

Thus we get Theorem 1 for the case $\delta>1/p-1$.

For the case $\delta=1/p-1$, we have, instead of (4.3) and (4.4),

$$\int_{-\pi}^{\pi}J_i(\theta)d\theta\leqq K(1-r)^{p\alpha}\Big(\log\frac{1}{1-r}\Big)^{-p\beta+1},\quad (i=1,2).$$

We can then prove the theorem for the second case, similarly as in the proof of the first case.

5. We shall prove Theorem 2. We consider the case $\delta>1/p$ only. We have, for this purpose,

$$\sum_{n=1}^{\infty}A_n^\delta\tau_n^\delta(\theta)z^n=\frac{ze^{i\theta}f'(ze^{i\theta})}{(1-z)^\delta},\quad (z=re^{it}).$$

Using the Hausdorff-Young theorem,

$$\Big\{\sum_{n=1}^{\infty}|A_n^\delta\tau_n^\delta(\theta)r^n|^{p'}\Big\}^{p/p'}\leqq K\int_{-\pi}^{\pi}|f'(re^{i(\theta+t)})|^p|1-re^{it}|^{-p\delta}dt.$$

But

$$\int_{-\pi}^{\pi}d\theta\int_{-\pi}^{\pi}\frac{|f'(re^{i(\theta+t)})|^p}{|1-re^{it}|^{p\delta}}dt\leqq K\int_{-\pi}^{\pi}\frac{dt}{[(1-r)^2+t^2]^{p\delta/2}}\int_{-\pi}^{\pi}|f'(re^{i(\theta+t)})|^p d\theta$$

$$\leqq K\Big[(1-r)^{-1+\alpha}\Big(\log\frac{1}{1-r}\Big)^{-\beta}\Big]^p\Big\{\int_0^{1-r}(1-r)^{-p\delta}dt+\int_{1-r}^{\pi}t^{-p\delta}dt\Big\}$$

$$\leqq K\Big[(1-r)^{-1+\alpha}\Big(\log\frac{1}{1-r}\Big)^{-\beta}\Big]^p(1-r)^{1-p\delta},\quad (\text{since } 1-p\delta<0),$$

$$\leqq K(1-r)^{-p+\alpha p+1-p\delta}\Big(\log\frac{1}{1-r}\Big)^{-\beta p}.$$

Using the above and taking $1-r=\pi/2^{\lambda+1}$, we get

$$\int_{-\pi}^{\pi}\Big\{\sum_{n=2^{\lambda-1}+1}^{2^\lambda}|\tau_n^\delta(\theta)|^{p'}\Big\}^{p/p'}d\theta\leqq K2^{\lambda(p-p\alpha-1)}\lambda^{-p\beta}$$

which corresponds to (4.5). Hence we can prove Theorem 2, similarly as in the proof of Theorem 1.

6. We shall now prove Theorem 3. By the same method, we have, for $\delta>1/p-1$,

$$\int_{-\pi}^{\pi}\left\{\sum_{n=2^{\lambda-1}+1}^{2^{\lambda}}|\sigma_n^{\delta}(\theta)-f(e^{i\theta})|^{p'}\right\}^{p/p'}d\theta\leqq K2^{\lambda(p-p\alpha-1)}.$$

Then, by the Hölder inequality,

$$\int_{-\pi}^{\pi}\left\{\sum_{n=2^{\lambda-1}+1}^{2^{\lambda}}|\sigma_n^{\delta}(\theta)-f(e^{i\theta})|^{k}\right\}d\theta$$

$$\leqq K2^{\lambda/q'}\left(\int_{-\pi}^{\pi}\left\{\sum_{n=2^{\lambda-1}+1}^{2^{\lambda}}|\sigma_n^{\delta}(\theta)-f(e^{i\theta})|^{p'}\right\}^{p/p'}d\theta\right)^{k/p}$$

$$\leqq K2^{\lambda(1-k\alpha)}.$$

Hence we have

$$\sum_{n=2}^{\infty}\int_{-\pi}^{\pi}\frac{|\sigma_n^{\delta}(\theta)-f(e^{i\theta})|^{k}}{n^{a}(\log n)^{b}}d\theta\leqq K\sum_{\lambda=1}^{\infty}\frac{1}{2^{\lambda a}\lambda^{b}}\int_{-\pi}^{\pi}\left\{\sum_{n=2^{\lambda-1}+1}^{2^{\lambda}}|\sigma_n^{\delta}(\theta)-f(e^{i\theta})|^{k}\right\}d\theta$$

$$\leqq K\sum_{\lambda=1}^{\infty}2^{\lambda(1-k\alpha-a)}\lambda^{-b}=K\sum_{\lambda=1}^{\infty}\lambda^{-b}<\infty,$$

since $1-k\alpha-a=0$ and $b>1$. Thus we get Theorem 3 for the case $\delta>1/p-1$.

We can also prove the other cases of Theorem 3 and Theorem 4.

7. Here we shall state a corollary.

Theorem 5. *Under the same assumption of Theorem 2, the series*

$$\sum n^{\Delta}c_n e^{ni\theta}$$

is summable $|C,\delta|$ *for almost all* θ, *where* $\Delta<1/k$.

This is the consequence of Theorem 2 and the following lemma due to H. C. Chow [1].

Lemma 2. *If* $0<\beta<1$ *and* $\{\lambda_n\}$ *is a sequence of positive numbers such that* $\Delta\lambda_n=\lambda_n-\lambda_{n+1}=O(\lambda_n/n)$ *and* λ_n/n *is non-increasing, and if the series* $\sum\lambda_n|\tau_n^{\beta}(\theta)|/n$ *is convergent, then the series* $\sum\lambda_n c_n e^{ni\theta}$ *is summable* $|C,\beta|$.

In fact, we have

$$\sum n^{\Delta}|\tau_n^{\delta}(\theta)|/n\leqq\left(\sum n^{k'(\Delta-1)}\right)^{1/k'}\left(\sum|\tau_n^{\delta}(\theta)|^{k}\right)^{1/k}<\infty,\ \text{a.e.},$$

since $k'(\Delta-1)<-1$. Thus we get Theorem 5.

From Theorem 4, we may also get a similar theorem.

References

[1] H. C. Chow: Theorems on power series and Fourier series, Proc. London Math. Soc., **1**, 206–216 (1951).

[2] G. H. Hardy and J. E. Littlewood: A convergence criterion for the Fourier series, Math. Zeits., **28**, 612–634 (1928).

[3] ——: Some properties of fractional integrals II, Math. Zeits., **34**, 403–439 (1932).

[4] S. Izumi: Some trigonometrical series. XIII, Proc. Japan Acad., **31**, 257–260 (1955).

[5] ——: Ditto. XVI, Proc. Japan Acad., **31**, 511–512 (1955).

[6] M. Kinukawa: On the convergence character of Fourier series. II, Proc. Japan Acad., **32**, 24–26 (1956).

[7] ——: Some strong summability of Fourier series, Proc. Japan Acad., **32**, 86–89 (1956).

[8] C. T. Loo: The absolute summability of power series, Duke Math. Jour., **10**, 271–276 (1943).