132. Note on Dimension Theory

By Jun-iti NAGATA
Department of Mathematics, Osaka City University
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Recently we have studied some necessary and sufficient conditions for n-dimensionality of general metric spaces.[1] The purpose of this note is to develop the previous results. That is, we shall give a generalization of our previous theorem concerning the relation between (Lebesgue's) dimension and covering and shall give some relations between metric function, length of covering and dimension. Moreover, we shall investigate embedding of n-dimensional metric spaces into products of 1-dimensional spaces on the foundation of our previous results.

All the topological spaces considered here are general metric spaces or metrizable spaces, and all the coverings are open unless otherwise mentioned.

Definition. A real valued function ρ of two points of a topological space R is a *non-Archimedean parametric* if

i) $\rho(x, y)\geqq 0$,

ii) $\rho(x, y)=\rho(y, x)$,

iii) $\{y \mid \rho(x, y)<\varepsilon\}$ is open for every $\varepsilon>0$,

iv) $\rho(x, y)\leqq \max[\rho(x, z), \rho(y, z)]$.

Theorem 1. *In order that* $\dim R\leqq n$ *for a metrizable space* R *it is necessary and sufficient that one can assign a metric* $\rho(x, y)$ *agreeing with the topology of* R *such that* $\rho(x, y)=\inf\{\rho_0(x, z_1)+\rho_0(z_1, z_2)+\cdots+\rho_0(z_p, y) \mid z_i\in R\}$, $\rho_0(x, y)=\min\{\rho_i(x, y) \mid i=1,\cdots,n+1\}$ *for some* $n+1$ *non-Archimedean parametrics* $\rho_i(x, y)$ $(i=1,\cdots, n+1)$.[2]

Proof. Necessity. Let $\dim R\leqq n$, then there exist $n+1$ 0-dimensional subspaces R_i such that $R=\bigcup_{i=1}^{n+1} R_i$ from the general decomposition theorem due to M. Katětov and to K. Morita.[3] We assign a metric $\rho'(x, y)$ of R such that $\rho'(x, y)\leqq 1$. Since R_i $(i=1,\cdots, n+1)$ are 0-dimensional, we get disjoint coverings[4] $\mathfrak{U}_m^i$ $(i=1,\cdots, n+1,\ m=1, 2\cdots)$

1) A theorem of dimension theory, Proc. Japan Acad., **32**, No. 3 (1956). On a relation between dimension and metrization, Proc. Japan Acad., **32**, No. 4 (1956).

2) This theorem contains, as a special case for $n=0$, Groot's theorem. See J. de Groot and H. de Vries: A note on non-Archimedean metrizations, Proceedings Koninkl. Nederl. Akademie van Wetenschappen, Ser. A, **58**, No. 2 (1955).

3) M. Katětov: On the dimension of non-separable spaces I, Czechoslovak Math. Jour., **2** (77) (1952). K. Morita: Normal families and dimension theory for metric spaces, Math. Annalen, **128** (1954).

4) We call a collection $\mathfrak{U}$ of open sets disjoint open collection if every intersection of two disjoint elements of $\mathfrak{U}$ is vacuous. If $\mathfrak{U}$ is a covering, it is called a disjoint covering. See " A theorem of dimension theory."

of R_i such that $\mathfrak{U}^i_{m+1}<\mathfrak{U}^i_m$, $\mathfrak{U}^i_m<\mathfrak{S}_m$ in R_i for $\mathfrak{S}_m=\{S_{1/2^m}(x)|x\in R\}$, where $S_{1/2^m}(x)=\{y\,|\,\rho'(x,y)<1/2^m\}$. We define open disjoint collections $\mathfrak{V}^i_m$ ($i=1,\cdots,n+1, m=0,1\cdots$) of R as follows. $\mathfrak{V}^i_0=\{R\}$. Let $\mathfrak{U}^i_{m+1}=\{\mathfrak{U}_\alpha|\alpha\in A\}$, then for every point $x\in R_i$ we can find $\alpha\in A$ such that $x\in U_\alpha$ and $\varepsilon(x)>0$ such that $S_{\varepsilon(x)}(x)\frown R_i\subseteq U_\alpha$, $S_{\varepsilon(x)}(x)\subseteq S_\alpha$ for some $S_\alpha\in\mathfrak{S}_{m+1}$ with $S_\alpha\supseteq U_\alpha$. We put $V_\alpha=\smile\{S_{\varepsilon(x)/2}(x)|x\in U_\alpha\frown R_i\}$ $(\subseteq S_\alpha)$. If $\mathfrak{V}^i_m$ is defined, then we define $\mathfrak{V}^i_{m+1}$ by $\mathfrak{V}^i_{m+1}=\mathfrak{V}^i_m\wedge\{V_\alpha|\alpha\in A\}$.[5] Then $\mathfrak{V}^i_m$ ($i=1,\cdots,n+1, m=0,1\cdots$) are obviously open disjoint collections such that $\mathfrak{V}^i_{m+1}<\mathfrak{V}^i_m<\mathfrak{S}_m$. Now we define a real valued function ρ_i of two points by $\rho_i(x,y)=\inf\{1/2^{m-1}|\,y\in S(x,\mathfrak{V}^i_m)\}$.[6] Then it is easily seen that ρ_i is a non-Archimedean parametric. It is also obvious that $\rho(x,y)=\inf\{\rho_0(x,z_1)+\cdots+\rho_0(z_p,y)|z_i\in R\}$ $(\rho_0(x,y)=\min\{\rho_i(x,y)|\,i=1,\cdots,n+1\})$ is a metric of R.

Sufficiency. Let $\rho(x,y)$ is a metric of R satisfying the condition, then we see easily that $R=\bigcup_{i=1}^{n+1}R_i$ for $R_i=\{x\,|\,\rho_i(x,x)=0\}$. To see this, we assume the existence of $x\in R$ such that $x\notin\bigcup_{i=1}^{n+1}R_i$. Then it must be $\rho_i(x,x)=\varepsilon_i>0$ ($i=1,\cdots,n+1$), and hence from the property of ρ_i, it holds $\rho_i(x,y)=\max[\rho_i(x,y),\rho_i(x,y)]\geqq\rho_i(x,x)=\varepsilon_i$ for every $y\in R$. Therefore $\rho_0(x,y)\geqq\min\varepsilon_i>0$, and hence $\rho(x,z)\geqq\min\varepsilon_i>0$ for every $z\in R$, which is a contradiction.

Putting $S^i_{1/m}(x)=\{y\,|\,\rho_i(x,y)<1/m\}$, we see easily from iv) that $S^i_{1/m}(x)\frown S^i_{1/m}(y)\neq\phi$ implies $S^i_{1/m}(x)=S^i_{1/m}(y)$. Hence $\mathfrak{U}^i_m=\{S^i_{1/m}(x)\frown R_i\,|\,x\in R_i\}$ ($m=1,2\cdots$) are open disjoint covering of R_i. Moreover, since $y\in S(x,\mathfrak{U}^i_m)$ implies $\rho(x,y)\leqq\rho_i(x,y)<1/m$, $\{\mathfrak{U}^i_m\,|\,m=1,2\cdots\}$ is an open basis of R_i. Thus we conclude $\dim R_i=0$.[7] This combining with $R=\bigcup_{i=1}^{n+1}R_i$ implies $\dim R\leqq n$ from the general decomposition theorem.

Next we proceed to generalize our previous theorem:

In order that a T_1-topological space R is a metrizable space with $\dim R\leqq n$ it is necessary and sufficient that there exists a sequence $\mathfrak{V}_1>\mathfrak{V}^*_2>\mathfrak{V}_2>\mathfrak{V}^*_3>\cdots$ of open coverings such that $S(p,\mathfrak{V}_m)$ $(m=1,2\cdots)$ is a nbd basis for each point p of R and such that each set of $\mathfrak{V}_{m+1}$ intersects at most $n+1$ sets of $\mathfrak{V}_m$.[8]

Theorem 2. *In order that a T_1-topological space R is a metrizable space with $\dim R\leqq n$ it is necessary and sufficient that there exists a sequence $\mathfrak{V}_1>\mathfrak{V}^*_2>\mathfrak{V}_2>\mathfrak{V}^*_3>\cdots$ of open coverings such that $S(p,\mathfrak{V}_m)$ $(m=1,2\cdots)$ is a nbd basis for each point p of R and such that order $\mathfrak{V}_m\leqq n+1$ $(m=1,2\cdots)$.*

5) $\mathfrak{P}\wedge\mathfrak{Q}=\{P\frown Q|P\in\mathfrak{P}, Q\in\mathfrak{Q}\}$ for open collections $\mathfrak{P},\mathfrak{Q}$. $\mathfrak{P}<\mathfrak{Q}$ for open collections $\mathfrak{P},\mathfrak{Q}$ means that $P\subseteq Q$ for every $P\in\mathfrak{P}$ and for some $Q\in\mathfrak{Q}$.

6) We use Tukey's notations. See convergence and uniformity in topology (1940).

7) See Morita: Loc. cit.

8) A theorem of dimension theory, Theorem 2.

Proof. Since the necessity is contained in the above theorem, we prove only the sufficiency. Let $\mathfrak{V}_1=\{V_\alpha\mid\alpha\in A\}$, then we define U_α by $U_\alpha=\smile\{V\mid S(V,\mathfrak{V}_2)\subseteq V_\alpha, V\in\mathfrak{V}_2\}$. $\mathfrak{U}_1=\{U_\alpha\mid\alpha\in A\}$ is an open covering of R such that $\mathfrak{U}_1>\mathfrak{V}_2$, and each set of $\mathfrak{V}_2$ intersects at most $n+1$ sets of $\mathfrak{U}_1$. Next, we define U_β by $U_\beta=\smile\{V\mid S(V,\mathfrak{V}_4)\subseteq V_\beta, V\in\mathfrak{V}_4\}$ for $\mathfrak{V}_3=\{V_\beta\mid\beta\in B\}$. Then for $\mathfrak{U}_3=\{U_\beta\mid\beta\in B\}$ it holds $\mathfrak{V}_4<\mathfrak{U}_3<\mathfrak{U}_3^*<\mathfrak{V}_3^*<\mathfrak{V}_2<\mathfrak{U}_1$. Since each set of $\mathfrak{V}_4$ intersects at most $n+1$ sets of $\mathfrak{U}_3$, we can repeat this process and get a sequence $\mathfrak{U}_1>\mathfrak{V}_2>\mathfrak{U}_3^*>\mathfrak{U}_3>\mathfrak{V}_4>\mathfrak{U}_5^*>\mathfrak{U}_5>\mathfrak{V}_6>\cdots$ of open coverings such that each set of $\mathfrak{V}_{2m}$ intersects at most $n+1$ sets of $\mathfrak{U}_{2m-1}$. Hence $\mathfrak{U}_1>\mathfrak{U}_3^*>\mathfrak{U}_3>\mathfrak{U}_5^*>\cdots$ is a sequence satisfying the condition of the above theorem, proving $\dim R\leqq n$.

Theorem 3. *Let $n=n_1+n_2+\cdots+n_k$ for non-negative integers n_i $(i=1,\cdots,k)$. If there exist sequences $\mathfrak{V}_{1i}>\mathfrak{V}_{2i}^*>\mathfrak{V}_{2i}>\mathfrak{V}_{3i}^*>\cdots$ $(i=1,\cdots,k)$ of open coverings of a T_1-space R such that order $\mathfrak{V}_{mi}\leqq n_i+1$ and such that $S(p,\mathfrak{V}_m)$ for $\mathfrak{V}_m=\bigwedge_{i=1}^{k}\mathfrak{V}_{mi}$ $(m=1,2\cdots)$ is a nbd basis of p, then R is a metrizable space with $\dim R\leqq n$ and can be embedded in a product of k metrizable spaces R_i $(i=1,\cdots,k)$ with $\dim R_i\leqq n_i$. (This theorem contains as a special case the sufficiency part of Theorem 2.)*

Proof. As in the proof of Theorem 2, we can select sequences $\mathfrak{U}_{1i}>\mathfrak{U}_{2i}^{**}>\mathfrak{U}_{2i}>\mathfrak{U}_{3i}^{**}>\cdots$ $(i=1,\cdots,k)$ of open coverings such that $S(p,\mathfrak{U}_{m+1i})$ intersects at most n_i+1 sets of $\mathfrak{U}_{mi}$ and such that $S(p,\mathfrak{U}_m)$ $(m=1,2\cdots)$ is a nbd basis of p for $\mathfrak{U}_m=\bigwedge_{i=1}^{k}\mathfrak{U}_{mi}$. Let $\mathfrak{U}_{mi}=\{U_\alpha\mid\alpha\in A\}$ for fixed m,i, then we put $V_\alpha=S(U_\alpha,\mathfrak{U}_{m+1i})$, $W_{1/2^{m-1}}=R$, $W_{1/2^m}=S(V_\alpha^c,\mathfrak{U}_{m+2i})$; $W_{1/2^m+1/2^{m+1}}=S(W_{1/2^m},\mathfrak{U}_{m+3i})$, $W_{1/2^{m+1}}=S(V_\alpha^c,\mathfrak{U}_{m+3i})$; $W_{1/2^m+1/2^{m+1}+1/2^{m+2}}=S(W_{1/2^m+1/2^{m+1}},\mathfrak{U}_{m+4i})$, $W_{1/2^m+1/2^{m+2}}=S(W_{1/2^m},\mathfrak{U}_{m+4i})$, $W_{1/2^{m+2}+1/2^{m+2}}=S(W_{1/2^{m+1}},\mathfrak{U}_{m+4i})$, $W_{1/2^{m+2}}=S(V_\alpha^c,\mathfrak{U}_{m+4i}),\cdots$. Defining $f_{\alpha mi}(x)=\inf\{r\mid x\in W_r\}$, we get continuous functions $f_\alpha(\alpha\in A)$ satisfying $f_{\alpha mi}(V_\alpha^c)=0$, $f_{\alpha mi}(U_\alpha)=1/2^{m-1}$.[9] Clearly, for every $\varepsilon>0$ there exists $l_i=l_i(\varepsilon)$ such that $y\in S(x,\mathfrak{U}_{l_i i})$ implies $|f_{\alpha mi}(x)-f_{\alpha mi}(y)|<\varepsilon$ $(\alpha\in A_{mi}, m=1,2\cdots)$. We consider a topological product $P_i=P\{I_\alpha\mid\alpha\in A_{mi}, m=1,2\cdots\}$ of $P_\alpha=\{x\mid 0\leqq x\leqq 1/2^{m-1}\}$ $(\alpha\in A_{mi})$. Then we define a mapping F_i of R into P_i by $F_i(x)=\{f_{\alpha mi}(x)\mid\alpha\in A_{mi}, m=1,2\cdots\}$ $(x\in R)$.

Now we proceed to prove that $F_i(R)$ $(\subseteq P_i)$ is a metrizable space with $\dim F_i(R)\leqq n_i$. Since $N_\alpha=F_i(R)\frown\{\{p_\alpha\}\mid p_\alpha>0\}$ $(\alpha\in A_{mi})$ are open and since $f_{\alpha mi}(U_\alpha)=1/2^m$, $\smile\{U_\alpha\mid\alpha\in A_{mi}\}=R$, $\mathfrak{N}_{mi}=\{N_\alpha\mid\alpha\in A_{mi}\}$ is an open covering of $F_i(R)$. Let us show order $\mathfrak{N}_{mi}\leqq n_i+1$. If $\bigcap_{j=1}^{h}N_{\alpha_j}\neq\phi$ and $\alpha_j\in A_{mi}$ $(j=1,\cdots,h)$, then we can choose $p=\{p_\alpha\}\in F_i(R)$ and $x\in R$ such that $p\in\bigcap_{j=1}^{h}N_{\alpha_j}$, $F_i(x)=p$. Since $f_{\alpha_j mi}(x)=p_{\alpha_j}>0$ $(j=1,\cdots,h)$, it holds $x\in V_{\alpha_j}$, $\alpha_j\in A_{mi}$ $(j=1,\cdots,h)$. On the other hand, since each $S(p,$

9) $f(V)=\alpha$ means $f(x)=\alpha$ $(x\in V)$.

$\mathfrak{U}_{m+1i})$ intersects at most n_i+1 sets of $\mathfrak{U}_{mi}$, we have order $\{V_\alpha=S(U_\alpha, \mathfrak{U}_{m+1i}) \mid \alpha\in A_{mi}\}\leqq n_i+1$. Therefore we have $h\leqq n_i+1$. This means order $\mathfrak{N}_{mi}\leqq n_i+1$. Moreover $\mathfrak{N}_{m+1i}<\mathfrak{N}_{mi}$, and $S(p, \mathfrak{N}_{mi})$ $(m=1, 2\cdots)$ is a nbd basis of each point p of $F_i(R)$; its proof is left to the reader. From Theorem 2 we can conclude the metrizability of $F_i(R)$ and $\dim F_i(R)\leqq n_i+1$.

Now we define a mapping $F(x)$ of R into $F_1(R)\times F_2(R)\times\cdots\times F_k(R)$ by $F(x)=(F_1(x),\cdots, F_k(x))\in F_1(R)\times\cdots\times F_k(R)$ $(x\in R)$. Then $F(x)$ is, as easily seen, a homeomorphic mapping and consequently R is homeomorphic with the subspace $F(R)$ of the product space $F_1(R)\times\cdots\times F_k(R)$ with $\dim F_i(R)\leqq n_i$ $(i=1,\cdots,k)$. From the general product theorem due to Katětov and Morita[10] we have $\dim R\leqq n_1+\cdots+n_k=n$.

Theorem 4. *Every metric space R with $\dim R\leqq n$ can be topologically imbedded in a topological product of $n+1$ at most 1-dimensional metric spaces.*

Proof. If $\dim R\leqq n$, then it is easily shown that we can assign a covering $\mathfrak{B}$ and open collections $\mathfrak{U}_i$ $(i=1,\cdots,n+1)$ for every covering $\mathfrak{U}$ of R such that $\mathfrak{B}<\bigvee_{i=1}^{n+1}\mathfrak{U}_i<\mathfrak{U}$ and such that each $S^2(p,\mathfrak{B})$ intersects at most one of sets belonging to $\mathfrak{U}_i$ for a fixed i. Because $R=\bigcup_{i=1}^{n+1}R_i$ for some R_i with $\dim R_i=0$, and hence there exists a disjoint covering $\mathfrak{B}_i$ of R such that $\mathfrak{B}_i<\mathfrak{U}$ in R_i. For every point x of R_i we denote by $\varepsilon(x)$ a positive number such that $S_{\varepsilon(x)}(x)\frown R_i\subseteq V_\alpha\in\mathfrak{B}_i$, $S_{\varepsilon(x)}(x)\subseteq U_\alpha\in\mathfrak{B}_i$ for U_α defined by V_α. Then $\mathfrak{B}'_i=\{\smile\{S_{\varepsilon(x)/2}(x)\mid x\in V_\alpha\}\mid V_\alpha\in\mathfrak{B}_i\}$ is an open collection of R with $\smile_{i=1}^{n+1}\mathfrak{B}'_i<\mathfrak{U}$. Selecting a covering $\mathfrak{W}$ with $\mathfrak{W}^*<\bigvee_{i=1}^{n+1}\mathfrak{B}'_i$, we can define an open collection $\mathfrak{U}_i$ by $\mathfrak{U}_i=\{\smile\{W\mid S(W,\mathfrak{W})\subseteq V'_\alpha\}\mid V'_\alpha\in\mathfrak{N}'_i\}$. It is easy to see that $\bigvee_{i=1}^{n+1}\mathfrak{U}_i$ covers R and that each set of $\mathfrak{W}$ intersects at most one of sets of $\mathfrak{U}_i$. Choosing a covering $\mathfrak{B}$ with $\mathfrak{B}^{**}<\mathfrak{W}$, we have open collections and a covering satisfying the required condition.

We denote by $\mathfrak{S}_1>\mathfrak{S}_2^*>\mathfrak{S}_2>\mathfrak{S}_3^*>\cdots$ a sequence of covering such that $S(p,\mathfrak{S}_m)$ $(m=1,2\cdots)$ is a nbd basis for each point of R, and take a covering $\mathfrak{B}$ and collections $\mathfrak{U}_{1i}$ $(i=1,\cdots,n+1)$ having the above property for $\mathfrak{S}_2$, *i.e.* $\mathfrak{B}<\bigvee_{i=1}^{n+1}\mathfrak{U}_{1i}<\mathfrak{S}_2$ and $S^2(p,\mathfrak{B})$ intersects at most one set of $\mathfrak{U}_{1i}$. Let $\mathfrak{U}_{1i}=\{U_\alpha\mid\alpha\in A\}$ and define $\mathfrak{N}_{1i}$ by $\mathfrak{N}_{1i}=\{S(U_\alpha,\mathfrak{B}),\ R-\smile_{\alpha\in A}\overline{U}_\alpha\mid\alpha\in A\}$ for a fixed i, then $\mathfrak{N}_{1i}$ is a covering with order $\leqq 2$. Moreover, $\bigwedge_{i=1}^{n+1}\mathfrak{N}_{1i}<\mathfrak{S}_1$ is obvious from $\mathfrak{S}_2^*<\mathfrak{S}_1$ and from that $\bigcup_{i=1}^{n+1}\mathfrak{U}_{1i}$ covers R.

Now we notice that every covering $\mathfrak{P}$ with order $\leqq 2$ has a locally finite star-refinement $\mathfrak{Q}'$ with order $\leqq 2$. To show this, we put

10) Loc. cit.

$\mathfrak{P}=\{P_\delta \mid \delta \in D\}$ and denote by $\mathfrak{P}'$ a star-refinement of $\mathfrak{P}$. Then $\mathfrak{M}=\{M_\delta = \smile\{P' \mid S(P', \mathfrak{P}) \subseteq P_\delta, P' \in \mathfrak{P}'\} \mid \delta \in D\}$ is a locally finite refinement of $\mathfrak{P}$ with order $\leqq 2$. We define an open set L_δ for every $\delta \in D$ such that $M_\delta - \underset{\delta \neq \delta' \in D}{\smile} M_{\delta'} \subseteq L_\delta \subseteq \bar{L}_\delta \subseteq M_\delta$ and put $Q_\delta = L_\delta - \underset{\delta' \neq \delta}{\smile} \bar{L}_{\delta'}$, $\mathfrak{Q}=\{Q_\delta, M_\alpha \frown M_\beta \mid \delta, \alpha, \beta \in D, \alpha \neq \beta\}$. It is easy to see that $\mathfrak{Q}$ is an open covering such that $\mathfrak{Q}^\Delta < \mathfrak{P}$, order $\mathfrak{Q} \leqq 2$. Repeating such a process we have a locally finite Δ-refinement $\mathfrak{Q}'$ of $\mathfrak{Q}$ with order $\leqq 2$. $\mathfrak{Q}'$ satisfies the required conditions.

To show the existence of sequences $\mathfrak{N}_{1i} > \mathfrak{N}_{2i}^* > \mathfrak{N}_{2i} > \mathfrak{N}_{3i}^* > \cdots$ $(i=1, \cdots, n+1)$ of coverings with order $\leqq 2$ such that $\overset{n+1}{\underset{i=1}{\wedge}} \mathfrak{N}_{mi} < \mathfrak{S}_m$, we assume the existence of such $\mathfrak{N}_{li}$ for $l \leqq m$. Then, there exists, from the above notice, a locally finite covering $\mathfrak{N}_i$ with order $\leqq 2$ such that $\mathfrak{N}_i^* < \mathfrak{N}_{mi}$. Next, we can select open collections $\mathfrak{P}_i$ $(i=1, \cdots, n+1)$ and a covering $\mathfrak{Q}$ such that $\mathfrak{M} \wedge \mathfrak{S}_{m+2} > \overset{n+1}{\underset{i=1}{\vee}} \mathfrak{P}_i > \mathfrak{Q}$ for $\mathfrak{M}$ with $\mathfrak{M}^{**} < \overset{m+1}{\underset{i=1}{\wedge}} \mathfrak{N}_i$ and such that each $S^2(p, \mathfrak{Q})$ intersects at most one of sets belonging to $\mathfrak{P}_i$ for a fixed i. We put $\mathfrak{P}_i=\{P_\beta \mid \beta \in B\}$, $\mathfrak{N}_i=\{N_\gamma \mid \gamma < \tau\}$ and denote by $\gamma(\beta)$ the first ordinal γ such that $S(P_\beta, \mathfrak{Q}) \subseteq N_\gamma \in \mathfrak{N}_i$ for $\beta \in B$. Then we define a covering $\mathfrak{N}_{m+1i}$ by $\mathfrak{N}_{m+1i}=\{K_\gamma, S(P_\beta, \mathfrak{Q}) \mid \gamma < \tau, \beta \in B\}$, where we put $K_\gamma = N_\gamma - \smile\{\bar{P}_\beta \mid \gamma = \gamma(\beta)\} \smile \{S(P_\beta, \mathfrak{Q}) \mid \gamma \neq \gamma(\beta)\}$. It follows easily that $\mathfrak{N}_{m+1i} < \mathfrak{N}_i$ and order $\mathfrak{N}_{m+1i} \leqq 2$; its proof is left to the reader. It holds, from the fact that $\overset{n+1}{\underset{i=1}{\vee}} \mathfrak{P}_i$ covers R, $\overset{n+1}{\underset{i=1}{\wedge}} \mathfrak{N}_{m+1i} < (\overset{n+1}{\underset{i=1}{\vee}} \mathfrak{P}_i)^* < \mathfrak{S}_{m+2}^* < \mathfrak{S}_{m+1}$. The formula $\mathfrak{N}_{m+1i} < \mathfrak{N}_i$ combining with $\mathfrak{N}_i^* < \mathfrak{N}_{mi}$ implies $\mathfrak{N}_{m+1i}^* < \mathfrak{N}_{mi}$. This completes the induction, and hence we get sequences $\mathfrak{N}_{1i} > \mathfrak{N}_{2i}^* > \mathfrak{N}_{2i} > \mathfrak{N}_{3i}^* > \cdots$ $(i=1, \cdots, n+1)$ such that $\overset{n+1}{\underset{i=1}{\wedge}} \mathfrak{N}_{mi} < \mathfrak{S}_m$, order $\mathfrak{N}_{mi} \leqq 2$. Hence we can imbed, from Theorem 3, R into a topological product of $n+1$ metrizable spaces R_i with $\dim R_i \leqq 1$.

Definition. We call a covering $\mathfrak{U}$ a *multiplicative covering* if every non-empty intersection $\bigcap_{i=1}^{k} U_i$ of elements U_i $(i=1, \cdots, k)$ of $\mathfrak{U}$ is an element of $\mathfrak{U}$.

Definition. Let n be the maximal number such that there exists a sequence $U_1 \supsetneqq U_2 \supsetneqq \cdots \supsetneqq U_n$ of elements of a multiplicative covering $\mathfrak{U}$, then n is called the *length* of $\mathfrak{U}$.[11]

Definition. We mean by the *rank* of an element U of a multiplicative covering $\mathfrak{U}$ the maximal number r such that there exists a sequence $U=U_1 \supsetneqq U_2 \supsetneqq \cdots \supsetneqq U_r$ of elements of $\mathfrak{U}$.

Theorem 5. *In order that a T_1-space R is a metrizable space with* $\dim \leqq n$ *it is necessary and sufficient that there exists a sequence*

11) The definitions and investigations on length of finite covering are due to P. Alexandroff and A. Kolmogoroff: Endliche Überdeckung topologisher Räume, Fun. Math., **26** (1936).

$\mathfrak{U}_1 > \mathfrak{U}_2^* > \mathfrak{U}_2 > \mathfrak{U}_3^* > \cdots$ *of multiplicative coverings with length* $\leqq n+1$ *such that* $S(p, \mathfrak{U}_m)$ $(m=1, 2\cdots)$ *is a nbd basis of* p.

Proof. Since the necessity is clear, we show only the sufficiency.

Let us assume the existence of a sequence satisfying the condition of the proposition. If we denote by $U_{r\alpha}(\alpha \in A_r)$ all the elements of $\mathfrak{U}_1$ with rank r, then $\mathfrak{U}_1 = \{U_{r\alpha} \mid \alpha \in A_r, r=1,\cdots,n+1\}$. We define $V_{r\alpha}^{(i)}$ $(i=1,\cdots,n+1)$ by $V_{r\alpha}^{(1)} = U_{r\alpha}$, $V_{r\alpha}^{(i)} = \{x \mid S(x, \mathfrak{U}_i) \subseteq V_{r\alpha}^{(i-1)}\}^\circ$ $(i=2, 3, \cdots, n+1)$.[12] It follows easily that $V_{r\alpha}^{(n+1)} \subseteq \cdots \subseteq V_{r\alpha}^{(2)} \subseteq V_{r\alpha}^{(1)} = U_{r\alpha}$ and $\mathfrak{U}_i < \{V_{r\alpha}^{(i)} \mid \alpha \in A_r, r=1,\cdots,n+1\}$ $(i=1,\cdots,n+1)$ and $S(V_{r\alpha}^{(i)}, \mathfrak{U}_i) \subseteq V_{r\alpha}^{(i-1)}$ $(i=2,\cdots,n+1)$. Next, we define $M_{i\alpha}$ $(i=1,\cdots,n+1)$ by $M_{1\alpha} = V_{1\alpha}^{(1)} = U_{1\alpha}$, $M_{i\alpha} = V_{i\alpha}^{(i)} - \overline{\smile\{S(V_{1\alpha}^{(i)}, \mathfrak{U}_{n+2}) \mid \alpha \in A_1\} \smile \{S(V_{2\alpha}^{(i)}, \mathfrak{U}_{n+2}) \mid \alpha \in A_2\} \smile \cdots \smile \{S(V_{i-1\alpha}^{(i)}, \mathfrak{U}_{n+2}) \mid \alpha \in A_{i-1}\}}$ $(i=2,\cdots,n+1)$. Let us show $\mathfrak{U}_{n+2} < \mathfrak{M}_1 = \{M_{r\alpha} \mid \alpha \in A_r, r=1,\cdots,n+1\}$. Since $\mathfrak{U}_{n+2} < \{V_{r\alpha}^{(r)} \mid \alpha \in A_r, r=1, 2,\cdots,n+1\}$, it is possible to find for every $U \in \mathfrak{U}_{n+2}$ the minimum number r such that $V_{r\alpha}^{(r)} \supseteq U$. If it holds $U \frown S(V_{k\alpha'}^{(r)}, \mathfrak{U}_{n+2}) \neq \phi$ for some k with $1 \leqq k \leqq r-1$, and for $\alpha \in A_k$, then we have, from $\mathfrak{U}_{n+2}^* < \mathfrak{U}_r$, $U \subseteq S(V_{k\alpha'}^{(r)}, \mathfrak{U}_r) \subseteq V_{k\alpha'}^{(r-1)} \subseteq V_{k\alpha'}^{(k)}$, which is a contradiction. Hence it must be $U \frown S(V_{k\alpha'}^{(r)}, \mathfrak{U}_{n+2}) = \phi$ $(1 \leqq k \leqq r-1, \alpha' \in A_k)$. This combining with $U \subseteq V_{r\alpha}^{(r)}$ implies $U \subseteq M_{r\alpha}$, proving $\mathfrak{U}_{n+2} < \mathfrak{M}_1$. It remains to prove order $\mathfrak{M}_1 \leqq n+1$. In the case $\alpha, \beta \in A_1$, $\alpha \neq \beta$ implies clearly $M_{1\alpha} \frown M_{1\beta} = U_{1\alpha} \frown U_{1\beta} = \phi$.

To show the same assertion for $r > 1$, we prove that $U_{r\alpha} \frown U_{r\beta} = U_{r'\gamma}$ for $\alpha, \beta \in A_r, \gamma \in A_{r'}$ generally implies $V_{r\alpha}^{(r)} \frown V_{r\beta}^{(r)} = V_{r'\gamma}^{(r)}$. First, $V_{r'\gamma}^{(2)} \subseteq V_{r\alpha}^{(2)} \frown V_{r\beta}^{(2)}$ is obvious. Conversely, there exist nbds $P(x)$, $Q(x)$ of $x \in V_{r\alpha}^{(2)} \frown V_{r\beta}^{(2)}$ such that $S(P(x), \mathfrak{U}_2) \subseteq U_{r\alpha}$, $S(Q(x), \mathfrak{U}_2) \subseteq U_{r\beta}$. Hence $S(P(x) \frown Q(x), \mathfrak{U}_2) \subseteq U_{r\alpha} \frown U_{r\beta} = U_{r'\gamma}$. This means $x \in V_{r'\gamma}^{(2)}$, proving $V_{r'\gamma}^{(2)} = V_{r\alpha}^{(2)} \frown V_{r\beta}^{(2)}$. Repeating this process, we conclude $V_{r'\gamma}^{(r)} = V_{r\alpha}^{(r)} \frown V_{r\beta}^{(r)}$. We return now to the proof of the assertion: $M_{r\alpha} \frown M_{r\beta} = \phi$ $(\alpha \neq \beta)$. Using the notice above, we have $M_{r\alpha} \frown M_{r\beta} \subseteq V_{r\alpha}^{(r)} \frown V_{r\beta}^{(r)} - \overline{\smile\{S(V_{1\alpha}^{(r)}, \mathfrak{U}_{n+2}) \mid \alpha \in A_1\} \smile \{S(V_{2\alpha}^{(r)}, \mathfrak{U}_{n+2}) \mid \alpha \in A_2\} \smile \cdots \smile \{S(V_{r-1\alpha}^{(r)}, \mathfrak{U}_{n+2}) \mid \alpha \in A_{r-1}\}} \subseteq V_{r\alpha}^{(r)} \frown V_{r\beta}^{(r)} - S(V_{r'\gamma}^{(r)}, \mathfrak{U}_{n+2}) = \phi$, where $U_{r\alpha} \frown U_{r\beta} = U_{r'\gamma}$ and consequently $r' < r$; hence order $\mathfrak{M}_1 \leqq n+1$. Repeating the same process, we get a sequence $\mathfrak{M}_m$ $(m = 1, 2\cdots)$ of coverings with order $\leqq n+1$ such that $\mathfrak{U}_{1+(m-1)(n+1)} > \mathfrak{M}_m > \mathfrak{U}_{1+m(n+1)}$. Therefore we have, from Theorem 2, $\dim R \leqq n$.

12) We denote by A° the interior of A.