130. On a Radical in a Semiring

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In our previous paper [2], we considered the structure space of maximal ideals of a commutative semiring, and K. Iséki [3], one of the present authors, considered some relations of two structure spaces of it. In this paper, we shall consider a new kind of ideals of a semiring A with 0 (for the definition, see our paper [2]). A similar theory of an associative ring was treated by L. Fuchs [1].

An element a of A is said to be a *left zerodivisor* if there is a $b(\neq 0)$ of A such that $ab=0$. Let $\mathfrak{A}$ be a two sided ideal, if every element of $\mathfrak{A}$ is a left zerodivisor, then $\mathfrak{A}$ is said to be *left zerodivisor*. In the sequel, by the term *ideal*, we mean a *two sided* ideal. An ideal is *maximal left zerodivisor* if there is no left zerodivisor ideal containing properly it. By Zorn's lemma, any left zerodivisor ideal is contained in a maximal left zerodivisor. Following L. Fuchs [1], we shall define a left zeroid ideal. If $\mathfrak{A}$ is an ideal and $\mathfrak{A}+\mathfrak{B}$ for every left zerodivisor ideal $\mathfrak{B}$ is a left zerodivisor, then $\mathfrak{A}$ is said to be a *left zeroid* ideal. Therefore we have the following propositions which are proved easily.

Proposition 1. The sum of two left zeroid ideals is a left zeroid ideal.

Proposition 2. The join of all left zeroid ideals is also a left zeroid ideal.

The left zeroid ideal stated in Proposition 2 is said to be the *left radical* of A which is denoted by $\mathfrak{R}^{(l)}$. Similarly, we can define right zerodivisor ideals, right zeroid ideals and the right radical $\mathfrak{R}^{(r)}$ of A. We shall define the radical $\mathfrak{R}$ of A as $\mathfrak{R}^{(l)}\frown\mathfrak{R}^{(r)}$.

If every element of an ideal $\mathfrak{A}$ is nilpotent, $\mathfrak{A}$ is said to be a *nil* ideal. Then any nil ideal $\mathfrak{N}$ is left zeroid and right zeroid.

Let b be an element of $\mathfrak{N}$, and let $\mathfrak{A}$ be a left zerodivisor ideal. Then there is some positive integer n such that $b^n=0$. For an element a of $\mathfrak{A}$, $(a+b)^n$ is in $\mathfrak{A}$. Hence there is an element $c(\neq 0)$ such that $(a+b)^nc=0$. Let m be the least positive integer such that $(a+b)^mc=0$, then we have $(a+b)^{m-1}c\neq 0$. Hence $(a+b)\times(a+b)^{m-1}c=0$ implies that $a+b$ is a left zerodivisor. Hence $\mathfrak{N}$ is a left zeroid ideal. Similarly we can prove that $\mathfrak{N}$ is a right zeroid ideal.

Therefore *the nil radical defined as the join of all nil ideals of A is contained in the radical $\mathfrak{R}$.*

On the left radical $\mathfrak{R}^{(l)}$, we have the following

Theorem 1. The left radical $\mathfrak{R}^{(l)}$ is the intersection of all maximal left zerodivisor ideals $\mathfrak{M}_\alpha$.

Proof. Let R be the intersection of all $\mathfrak{M}_\alpha$, and let $\mathfrak{A}$ be a left zerodivisor ideal of A, then there is a maximal left zerodivisor ideal $\mathfrak{M}_\alpha$ containing the ideal $\mathfrak{A}$. Since R is an ideal, $R+\mathfrak{A}\subseteq\mathfrak{M}_\alpha$ and then $R+\mathfrak{A}$ is a left zerodivisor ideal. Hence R is contained in $\mathfrak{R}^{(l)}$. On the other hand, if there is a maximal left zerodivisor ideal $\mathfrak{M}_\alpha$ not containing $\mathfrak{R}^{(l)}$, $\mathfrak{R}^{(l)}+\mathfrak{M}$ is not a left zerodivisor ideal. Therefore $\mathfrak{R}^{(l)}$ is not left zeroid ideal. Hence we have $\mathfrak{R}^{(l)}\subset\mathfrak{M}_\alpha$. The proof is complete.

Theorem 2. Any maximal left zerodivisor ideal is prime.

Since A is not necessarily commutative, a prime ideal means prime in McCoy sense [4]. Therefore we shall show that $\mathfrak{A}\mathfrak{B}\subset\mathfrak{M}$ for two ideals $\mathfrak{A}, \mathfrak{B}$ of A implies $\mathfrak{A}\subset\mathfrak{M}$ or $\mathfrak{B}\subset\mathfrak{M}$.

Proof. Let $\mathfrak{M}$ be a maximal left zerodivisor ideal and $\mathfrak{A}\mathfrak{B}\subset\mathfrak{M}$. Suppose that both two $\mathfrak{A}, \mathfrak{B}$ are not contained in $\mathfrak{M}$. By the maximality of $\mathfrak{M}$, $a+m_1$ and $b+m_2$, where $a\in\mathfrak{A}$, $b\in\mathfrak{B}$ and $m_1, m_2\in\mathfrak{M}$, are not left zerodivisor. From $\mathfrak{A}\mathfrak{B}\subset\mathfrak{M}$, $(a+m_1)(b+m_2)=ab+m(m\in\mathfrak{M})$ is contained in $\mathfrak{M}$, and there is an element $c(\neq 0)$ such that

$$(a+m_1)(b+m_2)c=(ab+m)c=0.$$

Since $a+m_1$ is not left zerodivisor, we have $(b+m_2)c=0$. This shows that $b+m_2$ is a left zerodivisor, which is a contradiction. Therefore $\mathfrak{A}\subset\mathfrak{M}$ or $\mathfrak{B}\subset\mathfrak{M}$.

From Theorem 1, we have

Theorem 3. The radical of A is the intersection of all maximal left zerodivisor and right zerodivisor ideals.

References

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