126. On Decomposition Spaces of Locally Compact Spaces

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1. Introduction. As is well known, any decomposition space[1] of a compact Hausdorff space is normal if it is Hausdorff. The following theorem is a generalization of this fact:[2]

Theorem 1. *Let a Hausdorff space Y be a decomposition space of a Hausdorff space X. If X is locally compact and has the Lindelöf property, then Y is paracompact and normal.*

Theorem 1 fails to be true if we replace the condition "has the Lindelöf property" by "is paracompact". This is seen from Theorem 2 below; further it will be shown in 5 below that a Hausdorff space which is obtained as a decomposition space of a locally compact, paracompact, Hausdorff space is not always regular.

Theorem 2. *A Hausdorff space X is obtained as the image of a locally compact, paracompact, Hausdorff space under an open continuous mapping if and only if X is locally compact.*

In [1, p. 70] P. Alexandroff and H. Hopf have stated that the existence of a regular, non-normal, Hausdorff space which is a decomposition space of a normal Hausdorff space remains unknown to them. Our Theorem 2 assures the existence of such a decomposition space[3] and settles this question, since there exists a non-normal, locally compact Hausdorff space. However, the following theorem will give a stronger result.

Theorem 3. *A Hausdorff space X is obtained as the image of a locally compact metric space under an open continuous mapping if and only if X is locally compact and locally metrizable.*

In the Euclidean plane, let E be the union of the line $x=0$ and the points $a_{nk}=\left(\frac{1}{n}, \frac{k}{n^2}\right)$, $n=1,2,\cdots$; $k=0, \pm 1, \pm 2,\cdots$. If the sets $T_n(y)$, $n=1,2,\cdots$; $-\infty<y<\infty$ and one-point sets $\{a_{nk}\}$, $n=1,2,\cdots$;

1) "Decomposition space" = "Zerlegungsraum" in the sense of [1, p. 63].

2) There exists a non-regular Hausdorff space which is the image, under an open continuous mapping, of a metric space which is a countable sum of compact sets; cf. [1, p. 70, Beispiel 2], where in line 15 from the bottom "$m=2,3,\cdots$" should be replaced by "$m=n, n+1,\cdots$".

3) If g is an open (or closed) continuous mapping of a T_1-space Z onto a T_1-space X, then X is homeomorphic to a decomposition space of Z associated with the decomposition $\{g^{-1}(x) \mid x \in X\}$ (cf. [1, p. 65] and [8]).

$k=0, \pm 1, \pm 2, \cdots$ are chosen as a basis of open sets where $T_n(y)=\left\{(u, v) \mid u \leqq \frac{1}{n}, |v-y| \leqq u\right\} \frown E$, then we have a space E described in [2, p. 116, ex. 4] and [3]. E is a non-normal, locally compact, locally metrizable, Hausdorff space. Hence it is seen from Theorem 3 that there exists a non-normal, locally compact, Hausdorff space which is the image of a locally compact metric space under an open continuous mapping.

As is well known, the space of all ordinals less than the first uncountable ordinal ω_1 with the order topology is a non-paracompact, completely normal, locally compact, locally metrizable, Hausdorff space. Therefore we see by Theorem 3 that there exists a non-paracompact, locally compact, normal Hausdorff space which is obtained as the image of a locally compact metric space under an open continuous mapping.

On the other hand, a paracompact Hausdorff space which is obtained as the image of a locally compact metric space under an open continuous mapping is necessarily metrizable.[4]

2. The classes $\mathfrak{S}$ and $\mathfrak{S}'$. A Hausdorff space X will be said to belong to *the class* $\mathfrak{S}$ (resp. $\mathfrak{S}'$) if there exists a closed covering (resp. a countable closed covering) $\mathfrak{M}$ of X such that every set of $\mathfrak{M}$ is compact and a subset K of X is closed if the intersection $K \frown M$ is closed for every set M of $\mathfrak{M}$.

Lemma 1. *Any locally compact Hausdorff space belongs to the class* $\mathfrak{S}$.

Proof. Let X be a locally compact Hausdorff space and $\{C_\alpha\}$ a family of compact sets of X such that $\{\text{Int}\, C_\alpha\}$ is a basis for open sets of X. Then $\{C_\alpha\}$ is clearly a closed covering of X. If K is a subset of X such that $K \frown C_\alpha$ is closed for each α, then K is easily shown to be closed. Hence X belongs to the class $\mathfrak{S}$.

Lemma 2. *A locally compact Hausdorff space X with the Lindelöf property belongs to the class* $\mathfrak{S}'$.

Proof. By [4, Theorem 10] X is paracompact and normal. Hence there exists a locally finite closed covering $\{A_i\}$ of X which consists of a countable number of compact sets. If $K \frown A_i$ is closed for each i, then $K=\smile_i (K \frown A_i)$ is closed since $\{A_i\}$ is locally finite. Hence X belongs to the class $\mathfrak{S}'$.

Lemma 3. *Let a Hausdorff space Y be a decomposition space of a Hausdorff space X. If X belongs to the class $\mathfrak{S}$ (resp. $\mathfrak{S}'$), then Y belongs also to the class $\mathfrak{S}$ (resp. $\mathfrak{S}'$).*

4) Among the images of a non-compact, locally compact, metric space under closed continuous mappings there exists a non-metrizable, paracompact, Hausdorff space (cf. [7]).

Proof. By assumption there exists a closed covering (resp. a countable closed covering) $\mathfrak{M}$ of X such that every set of $\mathfrak{M}$ is compact and a subset K of X is closed if $K \frown M$ is closed for each set M of $\mathfrak{M}$. Let f be the natural mapping of X onto Y, and put $\mathfrak{N}=\{N \mid N=f(M),\ M\in\mathfrak{M}\}$. Then each set N of $\mathfrak{N}$ is compact. Since Y is Hausdorff, $\mathfrak{N}$ is a closed covering (resp. a countable closed covering) of Y. Let F be a subset of Y such that $F\frown N$ is closed for each set N of $\mathfrak{N}$. Since $N=f(M)$, $M\in\mathfrak{M}$ and $f^{-1}(F\frown N)=f^{-1}(F)\frown f^{-1}(N)$, we have $f^{-1}(F)\frown M=f^{-1}(F\frown N)\frown M$. Hence $f^{-1}(F)\frown M$ is closed for each set M of $\mathfrak{M}$. Therefore $f^{-1}(F)$ is closed. Since Y is a decomposition space of X, F is a closed set of Y. This shows that Y belongs to the class $\mathfrak{S}$ (resp. $\mathfrak{S}'$).

Lemma 4. *A Hausdorff space X belonging to the class $\mathfrak{S}$ (resp. $\mathfrak{S}'$) is homeomorphic to a decomposition space of a locally compact, paracompact, Hausdorff space (resp. a locally compact Hausdorff space with the Lindelöf property).*

Proof. By assumption there exists a closed covering (resp. a countable closed covering) $\{A_\alpha \mid \alpha\in\Omega\}$ of X such that each A_α is compact and a subset K is closed if $K\frown A_\alpha$ is closed for each α. Let Z be a locally compact Hausdorff space with the following properties:
(1) Z is a union of compact sets C_α, $\alpha\in\Omega$;
(2) each C_α is open in Z and $C_\alpha\frown C_\beta=0$ for $\alpha\neq\beta$;
(3) for each α there exists a homeomorphism φ_α of C_α onto A_α.
The existence of such a space Z is clear. Z is paracompact (resp. has the Lindelöf property).

We define a mapping g of Z onto X by $g(z)=\varphi_\alpha(z)$ for $z\in C_\alpha$. Then g is clearly a continuous mapping of Z onto X. Let K be a subset of X such that $g^{-1}(K)$ is closed. Then, for each α, $g^{-1}(K)\frown C_\alpha$ is compact and hence $g(g^{-1}(K)\frown C_\alpha)$ is compact. Since $g(g^{-1}(K)\frown C_\alpha)=K\frown A_\alpha$, $K\frown A_\alpha$ is closed in X for each α. Therefore K is closed. This shows that X is homeomorphic to a decomposition space of Z associated with the decomposition $\{g^{-1}(x) \mid x\in X\}$.

From these lemmas we obtain immediately

Theorem 4. *The class of all Hausdorff spaces which are obtained as decomposition spaces of locally compact, paracompact, Hausdorff spaces is identical with the class $\mathfrak{S}$.*

Theorem 5. *The class of all Hausdorff spaces which are obtained as decomposition spaces of locally compact Hausdorff spaces with the Lindelöf property is identical with the class $\mathfrak{S}'$.*

3. Proof of Theorem 1. Theorem 1 is a direct consequence of Lemma 5 below in view of Theorem 5.

Lemma 5. *A Hausdorff space belonging to the class $\mathfrak{S}'$ is paracompact and normal.*

Proof. Since a compact Hausdorff space is paracompact and normal, Lemma 5 follows immediately from a result obtained in a previous paper [6, Corollary to Theorem 1].[5]

4. Proofs of Theorems 2 and 3. Let X be locally compact (resp. locally compact and locally metrizable). For each point p of X there exists an open neighbourhood U_0 such that $\overline{U}_0$ is compact. Then there exists a real-valued continuous function $f(x)$ such that $f(p)=1$ and $f(x)=0$ for $x \in X-U_0$. If we put $U=\{x \mid f(x)>0\}$, U is a countable sum of compact sets and hence U is paracompact by [4, Theorem 10]. Thus there exists an open covering $\{V_\alpha \mid \alpha \in \Omega\}$ of X such that each V_α is paracompact (resp. metrizable). Let Z be a locally compact Hausdorff space such that

(1)′ Z is a union of open sets W_α, $\alpha \in \Omega$;

(2)′ $W_\alpha \frown W_\beta = 0$ for $\alpha \neq \beta$;

(3)′ for each α there exists a homeomorphism φ_α of W_α onto V_α.

The existence of such a space Z is evident. Z is paracompact (resp. metrizable).

We define a mapping g of Z onto X by putting $g(z)=\varphi_\alpha(z)$ if $z \in W_\alpha$. Then g is clearly a continuous mapping of Z onto X. Let H be any open set of Z. Then $H \frown W_\alpha$ is an open set of W_α for each α. Hence $g(H \frown W_\alpha)=\varphi_\alpha(H \frown W_\alpha)$ is an open sets of V_α and consequently an open set of X for each α. Since $g(H)=\smile_\alpha g(H \frown W_\alpha)$, $g(H)$ is itself an open set of X. This shows that g is an open mapping.

Thus the "if" part of Theorem 2 and that of Theorem 3 are proved. The "only if" part of Theorem 2 is obvious.

To prove the "only if" part of Theorem 3, suppose that X is the image of a locally compact metric space Z under an open continuous mapping g. Then X is locally compact. For each point x of X we take a point z from $g^{-1}(x)$. Then z has an open neighbourhood W which is a countable sum of compact sets. The subspace W is a locally compact metric space with a countable basis. Hence $g(W)$ is locally compact and has a countable basis, and consequently $g(W)$ is metrizable. On the other hand, $g(W)$ is an open neighbourhood of x. Thus X is locally metrizable. This proves the "only if" part of Theorem 3.

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5) In April, 1955, E. Michael communicated to me that he had proved [6, Theorem 1] independently. His proof seems to proceed along the same line as my proof of [5, Theorem 2] with Tietze's extension theorem replaced by his extension theorem in his paper: Selection theorems for continuous functions, Proc. Inter. Math. Cong. Amsterdam (1954). A few months later T. Kando found also the same proof as Michael's independently.

Added in proof: Cf. E. Michael: Continuous selections, I, Ann. Math., **63**, 361–382 (1956).

5. A remark. Let X be a locally compact Hausdorff space which is not normal. Then there exist two closed sets A and B such that there exist no open sets U and V with $A \subset U$, $B \subset V$, $U \frown V = 0$. If we construct a decomposition space Y by contracting A to a point, then Y is a Hausdorff space which is not regular. Thus *a Hausdorff space which is obtained as a decomposition space of a locally compact, paracompact, Hausdorff space is not always regular.*

On the other hand, a T_1-space which is the image of a locally compact, paracompact Hausdorff space under a closed continuous mapping is always paracompact (cf. [7]).

References

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