125. On Closed Mappings

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1. Introduction. In a previous paper [6], S. Hanai and the author have dealt with the problem: "Under what condition will the image of a metric space under a closed continuous mapping be metrizable ?", and obtained the second part of the following theorem; this result, as M. Tsuda has called our attention, was also obtained by A. H. Stone and announced in [7].

Theorem 1. *Let X be a metric space and let a topological space Y be the image of X under a closed continuous mapping f. Then Y is paracompact and perfectly normal. Furthermore, Y is metrizable if and only if the boundary $\mathfrak{B}f^{-1}(y)$ of the inverse image $f^{-1}(y)$ is compact for every point y of Y.*

In the present note we shall deduce the first part of Theorem 1 as an immediate consequence of Theorem 3 below, and establish an analogous result for the case of locally compact spaces; namely we shall prove the following theorems.

Theorem 2. *Let X be a paracompact and locally compact Hausdorff space and let a topological space Y be the image of X under a closed continuous mapping f. Then Y is a paracompact Hausdorff space. Furthermore Y is locally compact if and only if the boundary $\mathfrak{B}f^{-1}(y)$ of the inverse image $f^{-1}(y)$ is compact for every point y of Y.*

Theorem 3. *Let X be a paracompact and perfectly normal space and let a topological space Y be the image of X under a closed continuous mapping f. Then Y is paracompact and perfectly normal.*

The second part of Theorem 2 is a direct consequence of Theorem 4 below.

Theorem 4. *Let f be a closed continuous mapping of a paracompact and locally compact Hausdorff space X onto another topological space Y. Denote by Y_0 [or Y_1] the set of all points y of Y such that $f^{-1}(y)$ [or $\mathfrak{B}f^{-1}(y)$] is not compact. Then we have $Y_1 \subset Y_0$ and*

(a) *Y_0 is a closed discrete subset of Y;*

(b) *$Y-Y_1$ is locally compact;*

(c) *the closure of any neighbourhood of y is not compact for every point y of Y_1.*

From Theorem 4 we obtain immediately

Corollary. *Under the assumption of Theorem 4 the mapping f admits of a factorization $f=f_2 \circ f_1$ such that*

(i) $f_1: X \to Z$ *is a closed continuous mapping onto* Z *and* $\{f_1^{-1}(z) \mid z \in Z_2\}$ *is a discrete collection where* Z_2 *is the set of points* z *such that* $f_1^{-1}(z)$ *contains at least two points;*
(ii) $f_2: Z \to Y$ *is a closed continuous mapping and* $f_2^{-1}(y)$ *is compact for every point* y *of* Y.

Furthermore we can prove

Theorem 5. *Let* f *be a closed continuous mapping of a paracompact and locally compact Hausdorff space* X *onto a locally compact space* Y. *Then* f *can be extended to a continuous mapping of* $\gamma(X)$ *onto* $\gamma(Y)$, *where* $\gamma(X)$ *and* $\gamma(Y)$ *mean the Freudenthal compactifications of* X *and* Y *respectively.*[1]

2. Proof of (a) of Theorem 4. It is sufficient to prove that $\{f^{-1}(y) \mid y \in Y_0\}$ is a discrete collection of closed sets in X. For this purpose we shall show that any compact set C intersects only a finite number of sets $f^{-1}(y)$, $y \in Y_0$. Suppose that there exists a countably infinite number of points x_i, $i=1,2,\cdots$ of X such that

$$x_i \in C \frown f^{-1}(y_i),\ y_i \in Y_0,\ i=1,2,\cdots;\ y_i \neq y_j \text{ for } i \neq j.$$

Since C is compact there exists a limit point x_0 of the set $\{x_i \mid i=1,2,\cdots\}$. We may assume that $f(x_0) \neq y_i$, $i=1,2,\cdots$; if $f(x_0)=y_i$ for some i, we have only to replace $\{x_j\}$ by $\{x_j \mid j \neq i\}$. Putting $y_0=f(x_0)$, we have

$$(1) \qquad y_0 \in Y_0;\ y_0 \neq y_i \text{ for } i=1,2,\cdots.$$

To prove (1) suppose that $y_0 \in Y - Y_0$. Then $f^{-1}(y_0)$ is compact. Since X is locally compact there exists an open set L such that $\overline{L}$ is compact and $f^{-1}(y_0) \subset L$. If we put $M = Y - f(X-L)$, then M is an open set of Y and $x_0 \in f^{-1}(y_0) \subset f^{-1}(M) \subset L$. The point x_0 is a limit point of $\{x_i\}$ and hence $x_i \in f^{-1}(M)$ for some i. Therefore for such i we have $f^{-1}(y_i) \subset f^{-1}(M) \subset L$. Thus $f^{-1}(y_i)$ must be compact but this contradicts the assumption that $y_i \in Y_0$. This proves (1).

By the assumption of the theorem X is paracompact and locally compact, and hence there exists a locally finite open covering $\{G_\alpha \mid \alpha \in \Omega\}$ of X such that $\overline{G}_\alpha$ is compact for each α. If we put

$$(2) \qquad \Gamma = \{\alpha \mid G_\alpha \frown f^{-1}(y_0) \neq 0\},$$

Γ is an infinite set, since $f^{-1}(y_0)$ is not compact. Let us put

$$(3) \qquad G = \smile \{G_\alpha \mid \alpha \in \Gamma\},\ V_0 = Y - f(X-G);$$

then V_0 is open and $f^{-1}(y_0) \subset f^{-1}(V_0) \subset G$.

The set of all points x_i which belong to $f^{-1}(V_0)$, since X is a T_1-space, consists of an infinite number of points; these points will be denoted by x_{k_i}, $i=1,2,\cdots$. Then x_0 is clearly a limit point of the set $\{x_{k_i}\}$. Therefore if we put $D = \{y_{k_i} \mid i=1,2,\cdots\}$ we have

$$(4) \qquad y_0 \in \overline{D} - D.$$

1) As for the Freudenthal compactifications, cf. [5].

Now we have $x_{k_i}\in f^{-1}(V_0)$ and hence $y_{k_i}\in V_0$. Thus

$$(5)\qquad f^{-1}(y_{k_i})\subset f^{-1}(V_0)\subset G,\ i=1,2,\cdots\ .$$

In view of (3) and (5) we can find points x'_{k_i} of X and elements α_i of Γ such that

$$(6)\qquad \begin{aligned} &x'_{k_1}\in f^{-1}(y_{k_1})\frown G_{\alpha_1},\\ &x'_{k_i}\in f^{-1}(y_{k_i})\frown (X-\bigcup_{j=1}^{i-1}G_{\alpha_j})\frown G_{\alpha_i},\ i=2,3,\cdots; \end{aligned}$$

indeed, since $f^{-1}(y_{k_i})$ is not compact, we have $f^{-1}(y_{k_i})\frown(X-\bigcup_{j=1}^{i-1}G_{\alpha_j})\neq 0$ for any finite number of sets $G_{\alpha_1},\cdots,G_{\alpha_{i-1}}$, and hence these x'_{k_i}, α_i can be found by induction.

Since $x'_{k_i}\in G_{\alpha_i}$, $\alpha_i\neq\alpha_j$ for $i\neq j$ and $\{G_\alpha\mid\alpha\in\Gamma\}$ is locally finite, the set $\{x'_{k_i}\mid i=1,2,\cdots\}$ is a closed subset of X. Therefore $D=\{y_{k_i}\}$ is closed in Y, since f is a closed map. However, (4) shows that D is not closed in Y. Thus we are led to a contradiction, and the assertion (a) in Theorem 4 is proved.

3. **Proof of (b) of Theorem 4** (cf. Hanai [3]). Let $y\in Y-Y_1$. Then $\mathfrak{B}f^{-1}(y)$ is compact. Since X is locally compact, there exists an open set L such that $\bar{L}$ is compact and $\mathfrak{B}f^{-1}(y)\subset L$. If we put $U=f^{-1}(y)\smile L$, $V=Y-f(X-U)$, then V is open in Y and $f^{-1}(y)\subset f^{-1}(V)\subset U$. Hence we have $\bar{V}\subset\overline{f(U)}=f(\bar{U})=f(\bar{L})\smile y$. Thus $\bar{V}$ is compact. This proves (b) of Theorem 4.

4. **Proof of (c) of Theorem 4.** Let $y_1\in Y_1$. Then $\mathfrak{B}f^{-1}(y_1)$ is not compact. According to (a) of Theorem 4 proved in 2 the set $F=\smile\{f^{-1}(y)\mid y\in Y_0-y_1\}$ is a closed set of X, and $F\frown f^{-1}(y_1)=0$. Hence if we put $V=Y-f(F)$, V is an open set of Y and $y_1\in V$.

Suppose that there exists a neighbourhood of y_1 whose closure is compact. Then there exists also an open neighbourhood V_1 of y_1 such that $\bar{V}_1$ is compact and $V_1\subset V$.

Since $\mathfrak{B}f^{-1}(y_1)\subset f^{-1}(V_1)$ and $\mathfrak{B}f^{-1}(y_1)$ is not compact and X is paracompact, there exists a locally finite collection $\{G_\alpha\mid\alpha\in\Gamma\}$ of open sets of X such that Γ is an infinite set and

$$(7)\qquad \mathfrak{B}f^{-1}(y_1)\subset\smile\{G_\alpha\mid\alpha\in\Gamma\},$$

$$(8)\qquad G_\alpha\subset f^{-1}(V_1)\ \text{for each}\ \alpha,$$

$$(9)\qquad G_\alpha\frown\mathfrak{B}f^{-1}(y_1)\neq 0\ \text{for each}\ \alpha.$$

In view of (9) we can take for each α a point x_α of X such that $x_\alpha\in(X-f^{-1}(y_1))\frown G_\alpha$. Since $\{G_\alpha\}$ is locally finite the set $A=\smile\{x_\alpha\mid\alpha\in\Gamma\}$ is a closed discrete set, and moreover A consists of infinitely many points.

On the other hand, the cardinal number of A is shown to be finite as follows. Since $\bar{V}_1$ is compact and $f(A)$ is discrete, $f(A)$ must be a finite set of points. By the construction of A, we have

$f(A) \subset V_1 - y_1 \subset Y - Y_0$. Hence $f^{-1}(y)$ is compact for every point y of $f(A)$ and consequently $A \frown f^{-1}(y)$ consists of a finite number of points since A is discrete. Therefore A must be a finite set of points. This is a contradiction. Thus the assertion (c) in Theorem 4 is proved.

5. Proof of Theorem 2. The second part of Theorem 2 follows readily from (b) and (c) of Theorem 4. To prove the first part we shall need the following lemmas.

Lemma 1. *Let X be a collectionwise normal space. If there exists a closed subset A of X such that A and every closed subste of X contained in $X-A$ are paracompact, then X is paracompact.*

This lemma follows readily from a theorem of C. H. Dowker [2, Lemma 1].[2)]

Lemma 2. *Let f be a closed continuous mapping of a paracompact normal space X onto another topological space Y and let Y_1 be the set of points y of Y such that $\mathfrak{B}f^{-1}(y)$ is not compact. If $\{f^{-1}(y) \mid y \in Y_1\}$ is a discrete collection in X, then Y is paracompact.*

Proof. Let F be any closed set of Y such that $F \subset Y - Y_1$. If we denote by g the partial map $f \mid f^{-1}(F)$, then g is a closed continuous mapping of $f^{-1}(F)$ onto F such that $\mathfrak{B}g^{-1}(y)$ is compact for every point y of F. Therefore F is paracompact by [6, Theorem 3]. Since Y_1 is a closed discrete set, Y_1 is paracompact. Moreover Y is collectionwise normal by [6, Theorem 3]. Therefore Y is paracompact by Lemma 1.

Now the first part of Theorem 2 is a direct consequence of Lemma 2.

6. Proof of Theorem 5. Theorem 5 follows from Lemma 3 below and Theorem 2 by an argument given in the proof of [5, Theorem 3].

Lemma 3. *Let f be a closed continuous mapping of a topological space X onto another topological space Y such that $\mathfrak{B}f^{-1}(y)$ is compact for every point y of Y. If A is a closed set of Y whose boundary $\mathfrak{B}A$ is compact, then $\mathfrak{B}f^{-1}(A)$ is compact.*

Proof. Since f is closed, we have $\mathfrak{B}f^{-1}(A) = f^{-1}(A) \frown \overline{X - f^{-1}(A)} \subset f^{-1}(A) \frown f^{-1}(\overline{Y-A}) = f^{-1}(\mathfrak{B}A)$. For $y \in A$, $\mathrm{Int}\, f^{-1}(y) \subset \mathrm{Int}\, f^{-1}(A)$ and hence $f^{-1}(y) \frown \mathfrak{B}f^{-1}(A) \subset \mathfrak{B}f^{-1}(y)$. Therefore if we denote by g the partial map $f \mid \mathfrak{B}f^{-1}(A)$, then g is a closed continuous map of

2) By a theorem of E. Michael [4, Theorem 1] and a theorem of Dowker mentioned above it can easily be shown that a collectionwise normal space is paracompact if it is a countable sum of closed sets each of which is paracompact. This is also proved by K. Nagami. This proposition and Lemma 1 fail to be valid if "collectionwise normal" is replaced by "normal"; cf. C. H. Dowker: Local dimension of normal spaces, Quart. J. Math., **6**, 101–120 (1955).

$\mathfrak{B}f^{-1}(A)$ onto $f(\mathfrak{B}f^{-1}(A))=K$ such that $g^{-1}(y)$ is compact for every point y of K. Since K is compact as a closed subset of $\mathfrak{B}A$, $\mathfrak{B}f^{-1}(A)$ is also compact. This proves Lemma 3.

7. Proof of Theorem 3. We note first that Y is collectionwise normal (cf. [6, Theorem 3]). It is also obvious that Y is perfectly normal.

Let $\{G_\alpha \mid \alpha<\Omega\}$ be any open covering of Y where α ranges over all ordinals less than a fixed ordinal Ω. Then $\{f^{-1}(G_\alpha) \mid \alpha<\Omega\}$ is an open covering of X. Since X is paracompact, there exists a locally finite closed covering $\{A_\alpha \mid \alpha<\Omega\}$ of X such that $A_\alpha \subset f^{-1}(G_\alpha)$ for each α. Since $\cup\{A_\gamma \mid \gamma<\alpha\}$ is a closed set of X and f is a closed map, the union $\cup\{f(A_\gamma) \mid \gamma<\alpha\}$ is closed in Y. As is remarked above Y is perfectly normal. Hence $f(A_\alpha)-\cup\{f(A_\gamma) \mid \gamma<\alpha\}$ is an F_σ-set of Y. Therefore there exists a countable number of closed sets $F_{\alpha i}$, $i=1,2,\cdots$ of Y such that

$$f(A_\alpha)-\cup\{f(A_\gamma) \mid \gamma<\alpha\}=\bigcup_{i=1}^{\infty} F_{\alpha i} \quad \text{for} \quad 1\leqq\alpha<\Omega.$$

Then we have clearly

$$F_{\alpha i} \frown F_{\beta j}=0 \quad \text{for} \quad 1\leqq\alpha<\beta<\Omega.$$

Let Γ be any subset of the set $\{\alpha \mid 1\leqq\alpha<\Omega\}$. Then $\{f^{-1}(F_{\alpha i}) \frown A_\alpha \mid \alpha\in\Gamma\}$ is a locally finite collection of closed sets of X and hence the union $\cup\{f^{-1}(F_{\alpha i})\frown A_\alpha \mid \alpha\in\Gamma\}$ is closed. Therefore $\cup\{F_{\alpha i} \mid \alpha\in\Gamma\}$ is closed, since $F_{\alpha i}\subset f(A_\alpha)$ and hence $f(f^{-1}(F_{\alpha i})\frown A_\alpha)=F_{\alpha i}$.

Thus for each $i=1,2,\cdots$ the family $|F_{\alpha i} \mid 1\leqq\alpha<\Omega|$ is a discrete collection of closed sets in Y. Since Y is collectionwise normal and

$$Y=\cup\{F_{\alpha i} \mid 1\leqq\alpha<\Omega,\ i=1,2,\cdots\}\cup f(A_0)$$

and $F_{\alpha i}\subset G_\alpha$, $f(A_0)\subset G_0$, by a theorem of R. H. Bing [1, Theorem 13] we can find a locally finite open covering of Y which is a refinement of $\{G_\alpha\}$. This proves Theorem 3.

References

[1] R. H. Bing: Metrization of topological spaces, Canadian J. Math., **13**, 175–186 (1951).

[2] C. H. Dowker: On a theorem of Hanner, Ark. för Mat., **2**, 307–313 (1952).

[3] S. Hanai: On closed mappings, Proc. Japan Acad., **30**, 285–288 (1954).

[4] E. Michael: A note on paracompact spaces, Proc. Amer. Math. Soc., **4**, 831–838 (1953).

[5] K. Morita: On images of an open interval under closed continuous mappings, Proc. Japan Acad., **32**, 15–19 (1956).

[6] K. Morita and S. Hanai: Closed mappings and metric spaces, Proc. Japan Acad., **32**, 10–14 (1956).

[7] A. H. Stone: Metrisability of decomposition spaces (Abstract), Bull. Amer. Math. Soc., **61**, 309 (1955).