154. *Note on Mapping Spaces*

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1. The set of all continuous mappings of a topological space X into another topological space Y is turned into a topological space by the compact-open topology; this topology is defined by selecting as a sub-basis for the open sets the family of sets $T(K,G)$ where K ranges over all the compact sets of X and G ranges over all the open sets of Y and $T(K,G)$ denotes the set of all continuous mappings f of X into Y such that $f(K)\subset G$. As usual we write Y^X for the mapping space.

Let X, Y be Hausdorff spaces and let Z be a topological space. With any continuous mapping f of $X\times Y$ into Z there is associated a mapping f^* from Y to the mapping space Z^X by the formula

(1) $$[f^*(y)](x)=f(x, y).$$

The correspondence $f\rightarrow f^*$ defines a one-to-one mapping

(2) $$\theta: Z^{X\times Y}\rightarrow (Z^X)^Y.$$

R. H. Fox [2] proved that θ is onto if either (i) X is locally compact or (ii) X and Y satisfy the first axiom of countability. It will be shown below (Theorem 1) that θ is always a homeomorphism into. Therefore θ is a homeomorphism onto in the above two cases (i) and (ii).[1] However, the case in which X is a CW-complex in the sense of J. H. C. Whitehead [5] and Y is a compact Hausdorff space seems to be not treated in the literature in spite of its importance in applications. In this note we shall prove that θ is a homeomorphism onto in this case also (Theorem 4).[2] This result will be obtained from a more general theorem (Theorem 2).

2. A Hausdorff space X will be said to have the weak topology with respect to compact sets in the wider sense if a subset A of X such that $A\frown K$ is closed for every compact set K of X is necessarily closed.[3] As is proved in [4], a Hausdorff space X has the weak topology with respect to compact sets in the wider sense if and only if X is obtained as a decomposition space of a locally compact, paracompact Hausdorff space.

1) M. G. Barratt [1, p. 81] has stated without proof that θ is a homeomorphism onto in case (i) with Y arbitrary and in case (ii) with $Y=I$ (the closed unit interval).

2) Thus the track group $(P, Q)^m(X, x_0; x_0)$ in the sense of Barratt [1] is isomorphic to the m-th homotopy group of the mapping space $(X, x_0)^{(P,Q)}$ with the base point $P\rightarrow x_0$ in case P is a Hausdorff space which is locally compact or satisfies the first axiom of countability, or a CW-complex.

3) A T_1-space having the weak topology with respect to compact sets in the sense of [3] is nothing but a discrete space.

Lemma 1. *A Hausdorff space X has the weak topology with respect to compact sets in the wider sense if (a) X is locally compact, or (b) X is a CW-complex in the sense of J. H. C. Whitehead, or (c) X satisfies the first axiom of countability.*

Proof. The lemma is obvious for the cases (a) and (b) (cf. [4, 5]). Suppose that X satisfies the first axiom of countability. Let A be a non-closed set of X. Then there exists a point x_0 such that $x_0 \notin A$ and $x_0 \in \bar{A}$. Let $\{V_n(x_0) \mid n=1, 2, \cdots\}$ be a basis of neighbourhoods at x_0. If we take points x_n, $n=1, 2, \cdots$ successively so that $x_n \in (V_n(x_0) - \bigcup_{i=1}^{n-1} x_i) \frown A$, $n=2, 3, \cdots$, then the set $K=\{x_i \mid i=0, 1, 2, \cdots\}$ is compact since X is Hausdorff, and $A \frown K$ is not closed. Therefore X has the weak topology with respect to compact sets in the wider sense.

Lemma 2. *If X is a Hausdorff space having the weak topology with respect to compact sets in the wider sense and Y is a locally compact Hausdorff space, then $X \times Y$ has the weak topology with respect to compact sets in the wider sense.*[4]

Proof. Let A be a subset of $X \times Y$ such that $A \frown (K \times L)$ is closed for every compact set K of X and for every compact set L of Y. Then for every compact set K of X, $A \frown (K \times Y)$ is closed in $K \times Y$, since $K \times Y$ is locally compact and every compact set of $K \times Y$ is contained in $K \times L$ for some compact set L of Y. Therefore $A \frown (K \times Y)$ is closed for every compact set K of X and hence A is closed as is seen from the proof of [3, Lemma 3].

3. For the sake of completeness we shall state two lemmas due to Fox [2].

Lemma 3. *If $f \in Z^{X \times Y}$, then $f^* \in (Z^X)^Y$.*

Proof. Let K be a compact set of X and G an open set of Z. If y is a point in $f^{*-1}(T(K, G))$, then $K \times y \subset f^{-1}(G)$ and hence there exists an open set V such that $y \in V$, $K \times V \subset f^{-1}(G)$ since K is compact. This shows that $V \subset f^{*-1}(T(K, G))$. Hence $f^* \in (Z^X)^Y$.

Lemma 4. *Let X be a Hausdorff space. If a subspace A of X is locally compact, then the mapping μ of $A \times Z^X$ into Z defined by $\mu(x, f) = f(x)$ is continuous.*

Proof. Let G be any open set containing $f_0(x_0)$ where $x_0 \in A$, $f_0 \in Z^X$. Since f_0 is continuous, $f_0^{-1}(G)$ is open and hence there exists an open set V of X such that $x_0 \in V$, $\overline{V \frown A} \frown A \subset f_0^{-1}(G)$ and $\overline{V \frown A} \frown A$ is compact. If $x \in V \frown A$, $f \in T(\overline{V \frown A} \frown A, G)$, then $f(x) \in G$. Hence μ is continuous.

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4) The topological product of two spaces having the weak topology with respect to compact sets in the wider sense has not always the weak topology with respect to compact sets in the wider sense.

4. Theorem 1. *If X and Y are Hausdorff spaces, then $\theta: Z^{X\times Y} \to (Z^X)^Y$ is a homeomorphism into.*

Proof. By Lemma 3 $f\in Z^{X\times Y}$ implies $\theta(f)\in(Z^X)^Y$. θ is clearly one-to-one. Let K be a compact set of X, L a compact set of Y and G an open set of Z. Then we have clearly

$$\theta(T(K\times L, G))=T(L, T(K, G))\frown\theta(Z^{X\times Y}). \tag{3}$$

To prove the continuity of θ, suppose that $f\in Z^{X\times Y}$ and

$$\theta(f)=f^*\in\bigcap_{i=1}^{m} T(L_i, G_i) \tag{4}$$

where L_i are compact sets of Y and G_i are open sets of Z^X. Then we have $f^*(L_i)\subset G_i$, $i=1,2,\cdots,m$, and hence for each i there exist a finite number of compact sets K_{ijk} of X and a finite number of open sets H_{ijk} of Z $(i=1,2,\cdots,m;\ j=1,2,\cdots,r(i);\ k=1,2,\cdots,s(i,j))$ such that

$$f^*(L_i)\subset\bigcup_{j=1}^{r(i)}\left[\bigcap_{k=1}^{s(i,j)} T(K_{ijk}, H_{ijk})\right]\subset G_i,\ i=1,\cdots,m, \tag{5}$$

since $f^*(L_i)$ is compact. From (5) we have

$$L_i\subset\bigcup_{j=1}^{r(i)} f^{*-1}\left(\bigcap_{k=1}^{s(i,j)} T(K_{ijk}, H_{ijk})\right).$$

Since L_i is compact and Y is Hausdorff, the subspace L_i is normal. Hence there exist compact sets L_{ij}, $j=1,\cdots,r(i)$, such that

$$L_i=\bigcup_{j=1}^{r(i)} L_{ij},\quad L_{ij}\subset f^{*-1}\left(\bigcap_{k=1}^{s(i,j)} T(K_{ijk}, H_{ijk})\right),\quad j=1,\cdots,r(i).$$

Thus we have

$$\begin{aligned} f^*&\in\bigcap_{i=1}^{m}\bigcap_{j=1}^{r(i)} T\left(L_{ij}, \bigcap_{k=1}^{s(i,j)} T(K_{ijk}, H_{ijk})\right)\\ &=\bigcap_{i=1}^{m}\bigcap_{j=1}^{r(i)}\bigcap_{k=1}^{s(i,j)} T(L_{ij}, T(K_{ijk}, H_{ijk}))\subset\bigcap_{i=1}^{m} T(L_i, G_i).\end{aligned}$$

Therefore in virtue of (3) we see that θ is continuous.

To prove that θ^{-1} is continuous, suppose that $f\in Z^{X\times Y}$ and

$$f\in\bigcap_{i=1}^{m} T(A_i, H_i) \tag{6}$$

where A_i are compact sets of $X\times Y$ and H_i are open sets of Z. Then for each i we have $A_i\subset f^{-1}(H_i)$. Since each A_i is compact there exist a finite number of open sets U_{ij} of X and open sets V_{ij} of Y, $j=1,2,\cdots,s(i)$, such that

$$A_i\subset\bigcup_{j=1}^{s(i)}(U_{ij}\times V_{ij});\quad U_{ij}\times V_{ij}\subset f^{-1}(H_i),\ j=1,\cdots,s(i).$$

Since the subspace A_i is normal, there exist a finite number of compact sets A_{ij}, $j=1,\cdots,s(i)$, such that

$$A_i=\bigcup_{j=1}^{s(i)} A_{ij};\ A_{ij}\subset U_{ij}\times V_{ij},\ j=1,\cdots,s(i).$$

If we put $B_{ij}=\{x\mid(x,y)\in A_{ij}\}$, $C_{ij}=\{y\mid(x,y)\in A_{ij}\}$, then B_{ij}, C_{ij} are compact and we have clearly

$$A_{ij} \subset B_{ij} \times C_{ij} \subset U_{ij} \times V_{ij} \subset f^{-1}(H_i).$$

Therefore we have

$$f \in \bigcap_{i=1}^{m} \bigcap_{j=1}^{s(i)} T(B_{ij} \times C_{ij}, H_i) \subset \bigcap_{i=1}^{m} T(A_i, H_i).$$

Hence we see by (3) that θ^{-1} is continuous. This completes the proof of Theorem 1.

5. Now we shall prove

Theorem 2. *If $X \times Y$ is a Hausdorff space having the weak topology with respect to compact sets in the wider sense, then the mapping $\theta: Z^{X \times Y} \to (Z^X)^Y$ is a homeomorphism onto.*

Proof. In view of Theorem 1 it is sufficient to prove that θ is onto. Let g be any continuous mapping of Y into Z^X, i.e., $g \in (Z^X)^Y$. We define a mapping $\tilde{g}$ from $X \times Y$ to Z by $\tilde{g}(x, y) = [g(y)]\ (x)$ where $x \in X$, $y \in Y$. We shall prove that $\tilde{g} \in Z^{X \times Y}$.

Let K be any compact set of X. Then the partial mapping $\tilde{g} \mid K \times Y$ is expressed as the composite of two mappings λ and μ:

$\lambda: K \times Y \to K \times Z^X$, $\mu: K \times Z^X \to Z$ where $\lambda(x, y) = (x, g(y))$, $\mu(x, f) = f(x)$ for $x \in K$, $y \in Y$, $f \in Z^X$. Since $g \in (Z^X)^Y$, λ is continuous. Lemma 4 shows that μ is continuous. Hence $\tilde{g} \mid K \times Y$ is continuous.

Since every compact set of $X \times Y$ is contained in a set of the form $K \times L$ with a compact set K of X and a compact set L of Y, $\tilde{g} \mid A$ is continuous for every compact set A of $X \times Y$. By assumption $X \times Y$ has the weak topology with respect to compact sets in the wider sense, and hence $\tilde{g}$ is continuous over $X \times Y$. Therefore $\tilde{g} \in Z^{X \times Y}$. This proves that θ is onto.

The conclusion of the first part of the above proof remains true if K is locally compact. Hence we have

Theorem 3.[5] *If X and Y are Hausdorff spaces and X is locally compact, then $\theta: Z^{X \times Y} \to (Z^X)^Y$ is a homeomorphism onto.*

In virtue of Lemmas 1 and 2 we obtain the following theorems as corollaries to Theorem 2.

Theorem 4. *If X is a Hausdorff space having the weak topology with respect to compact sets in the wider sense (for instance, if X is a CW-complex) and Y is a locally compact Hausdorff space, then $\theta: Z^{X \times Y} \to (Z^X)^Y$ is a homeomorphism onto.*

Theorem 5.[5] *If X and Y are Hausdorff spaces satisfying the first axiom of countability, then $\theta: Z^{X \times Y} \to (Z^X)^Y$ is a homeomorphism onto.*

Theorem 2 is clearly generalized to the case of maps of pairs of spaces.

Theorem 6. *Let $X \times Y$ be a Hausdorff space having the weak topology with respect to compact sets in the wider sense, and let*

5) Cf. Fox [2], Barratt [1, p. 81].

$X_0 \subset X$, $Y_0 \subset Y$, $Z_0 \subset Z$. *Then* θ *induces a homeomorphism onto*:

$$\theta_0 : (Z, Z_0)^{(X \times Y,\ X_0 \times Y \cup X \times Y_0)} \to ((Z, Z_0)^{(X,\ X_0)},\ Z_0^X)^{(Y,\ Y_0)}.$$

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