167. A Remark on Fundamental Exact Sequences in Cohomology of Finite Groups

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The purpose of the present short note is to establish two exact sequences, and their duals, which form a bridge between the well-known series of so-called fundamental exact sequences in cohomology and homology of finite groups, and one of which has been made use of, if not in a way of absolute necessity, in a recent note [7] by the writer. Thus, we prove: Let G be a finite group, H an invariant subgroup of G, and M a G-module. Then the sequence

$$(2_0')\qquad 0\longleftarrow H^0(G/H, M^H)\xleftarrow{\varphi'} H^0(G,M)\xleftarrow{\iota} H^0(H,M)_G \xleftarrow{\tau'} H^{-1}(G/H, M^H)\xleftarrow{\varphi'} H^{-1}(G,M)$$

is exact.[1] Further, if $H^0(H,M)=0$, then the sequence

$$(2_1')\qquad 0\longleftarrow H^{-1}(G/H, M^H)\xleftarrow{\varphi'} H^{-1}(G,M)\xleftarrow{\iota} H^{-1}(H,M)_G \xleftarrow{\tau'} H^{-2}(G/H, M)\xleftarrow{\varphi'} H^{-2}(G,M)$$

is exact.[2] Dually, the sequence

$$(1_{-1}')\qquad 0\longrightarrow H^{-1}(G/H, M_H)\xrightarrow{\lambda'} H^{-1}(G,M)\xrightarrow{\rho} H^{-1}(H,M)^G \xrightarrow{\tau'} H^0(G/H, M_H)\xrightarrow{\lambda'} H^0(G,M)$$

is exact. If $H^{-1}(H,M)=0$, then the sequence

$$(1')\qquad 0\longrightarrow H^0(G/H, M_H)\xrightarrow{\lambda'} H^0(G,M)\xrightarrow{\rho} H^0(H,M)^G \xrightarrow{\tau'} H^1(G/H, M_H)\xrightarrow{\lambda'} H^1(G,M)$$

is exact. The significance of the maps in these sequences will be given in the sequel.

To begin with, we consider a not necessarily finite group G and an invariant subgroup H of G. Let M be a G-module. Then (Hochschild-Nakayama [5], Hochschild-Serre [6]):

I. If $m\geqq 1$ and if $H^n(H,M)=0$ for $n=1,2,\cdots, m-1$, then the sequence of cohomology groups

$$(1)\qquad 0\longrightarrow H^m(G/H, M^H)\xrightarrow{\lambda} H^m(G,M)\xrightarrow{\rho} H^m(H,M) \xrightarrow{\tau} H^{m+1}(G/H, M^H)\xrightarrow{\lambda} H^{m+1}(G,M)$$

is exact, where λ is a lifting map, ρ is a restriction map, and τ is a so-called transgression; that the transgression maps precisely

1) The first half of this exact sequence has been given in Artin-Tate [1].

2) The first half of this exact sequence has independently been obtained by Y. Kawada.

$H^m(H,M)^G$ into $H^{m+1}(G/H, M^H)$ (and not into its proper residue-group) depends on the hypothesis $H^n(H,M)=0$ $(n=1,\cdots, m-1)$. Dually

II. If $m\geqq 1$ and if $H_n(H,M)=0$ for $n=1,2,\cdots, m-1$, then the sequence of homology groups

$$(2)\qquad 0\longleftarrow H_m(G/H, M_H)\xleftarrow{\varphi} H_m(G,M)\xleftarrow{\iota} H_m(H,M)_G \xleftarrow{\tau} H_{m+1}(G/H, M_H)\xleftarrow{\varphi} H_{m+1}(G,M)$$

is exact, where τ is a transgression while ι and φ are defined as follows and may be called injection and residuation respectively: the element of $H_n(H,M)_G$ represented by the (n-homology) class $\in H_n(H,M)$ of an n-cycle $\sum_{\alpha_1,\cdots,\alpha_n\in H}(\alpha_1,\cdots,\alpha_n)\otimes u(\alpha_1,\cdots,\alpha_n)(u(\alpha_1,\cdots,\alpha_n)\in M)$ on H in M is mapped by ι to the class $\in H_n(G,M)$ of the n-cycle on G represented by the same expression, i.e. the n-cycle $\sum_{\sigma_1,\cdots,\sigma_n\in G}(\sigma_1,\cdots,\sigma_n)\otimes v(\sigma_1,\cdots,\sigma_n)$ where $v(\sigma_1,\cdots,\sigma_n)=u(\sigma_1,\cdots,\sigma_n)$ or 0 according as all σ_i belong to H or not; the class $\in H_n(G,M)$ of an n-cycle $\sum_{\sigma_1,\cdots,\sigma_n\in G}(\sigma_1,\cdots,\sigma_n)\otimes u(\sigma_1,\cdots,\sigma_n)$ on G in M is mapped by φ on the class $\in H_n(G/H, M_H)$ of the n-cycle $\sum_{\sigma_1,\cdots,\sigma_n\in G}(\bar\sigma_1,\cdots,\bar\sigma_n)\otimes \bar u(\sigma_1,\cdots,\sigma_n)$ on G/H in M_H where $\bar\sigma_i$ denotes the coset of σ_i modulo H and $\bar u(\sigma_1,\cdots,\sigma_n)$ denotes the image of $u(\sigma_1,\cdots,\sigma_n)$ by the canonical homomorphism $M\longrightarrow M_H$. Our terminology "injection" is in accord with Chevalley [2]; it corresponds to "transfer" in Artin-Tate [1] and Eckmann [3]. That the transgression τ maps $H_{m+1}(G/H, M_H)$ (and not its proper subgroup) into $H_m(H,M)_G$ depends on the hypothesis $H_n(H,M)=0$ $(n=1,\cdots, m-1)$.

If in particular G is finite, then (2) implies:

II′. If $m\geqq 2$ and if $H^{-n}(H,M)=0$ for $n=0,1,\cdots, m-1$, then the sequences of negative-dimensional cohomology groups

$$(2')\qquad 0\longleftarrow H^{-m}(G/H, M^H)\xleftarrow{\varphi'} H^{-m}(G,M)\xleftarrow{\iota} H^{-m}(H,M)_G \xleftarrow{\tau'} H^{-(m+1)}(G/H, M^H)\xleftarrow{\varphi'} H^{-(m+1)}(G,M)$$

is exact, where φ', a modified residuation, maps the class $\in H^{-n}(G,M)$ $(n\geqq 1)$ of an $(n-1)$-cycle $\sum_{\sigma_1,\cdots,\sigma_n\in G}(\sigma_1,\cdots,\sigma_{n-1})\otimes u(\sigma_1,\cdots,\sigma_{n-1})$ on G in M to the class $\in H^{-n}(G/H, M^H)$ of the $(n-1)$-cycle $\sum_{\sigma_1,\cdots,\sigma_n\in G}(\bar\sigma_1,\cdots,\bar\sigma_n)\otimes S_H u(\sigma_1,\cdots,\sigma_{n-1})$ on G/H in M^H, S_H being the trace map with respect to H, and where τ', a modified transgression, is defined by the transgression τ following the inverse of the isomorphism $H^{-n}(G/H, M_H)\longrightarrow H^{-n}(G/H, M^H)$ induced by the trace map $S_H: M_H\longrightarrow M^H$; indeed, our assumption includes $H^0(H,M)=H^{-1}(H,M)=0$ which implies that the G- (or G/H-)homomorphism $M_H\longrightarrow M^H$ effected by S_H is an isomorphism.

Now, we contend that II′ may be supplemented with two predecessors given in ($2'_0$), ($2'_1$) at the opening. The maps in ($2'_1$) except τ' are explained in the same manner as in II′, while the map τ' in ($2'_1$) is defined as follows under the assumption $H^0(H,M)=0$: Let

$\{\rho\}$ be a representative system of H-cosets in G, and let $\sum_{\rho\in\{\rho\}}(\bar\rho)\otimes u(\rho)$ $(\bar\rho=(\rho \bmod H))$ be a 1-cycle on G/H in M^H. By the assumption $H^0(H, M)=0$, take $v(\bar\rho)\in M$ for each ρ such that $S_H(v(\bar\rho))=u(\rho)$. Then τ' maps the class $\in H^{-2}(G/H, M^H)$ of our 1-cycle to the element of $H^{-1}(H, M)_G$ represented by the class $\in H^{-1}(H, M)$ of the 0-cycle $\sum_\rho(\rho^{-1}-1)v(\bar\rho)$; that the definition is independent of the choice of $v(\bar\rho)$ can readily be seen. As for $(2_0')$, the first φ', in $(2_0')$, is induced by the identity map $M^G\longrightarrow(M^H)^{G/H}$, while ι is induced by the trace map $S_{G/H}: M^H\longrightarrow M^G$ and is essentially the injection in the sense of Chevalley [2], or the transfer of Artin-Tate [1]. Further, τ' in $(2_0')$ sends the class $\in H^{-1}(G/H, M^H)$ of the 0-cycle u $(u\in M^H,\ S_{G/H}(u)=0)$ on G/H in M^H to the element of $H^0(H, M)_G$ represented by the class $\in H^0(H, M)$ of the 0-cocycle u on H in M. The second φ' in $(2_0')$ is defined in the same manner as in II′, i.e. by the residuation φ in combination with S_H. So, $(2_0')$ is nothing but

$$0\longleftarrow \frac{M^G}{S_{G/H}(M^H)}\xleftarrow{\varphi',1}\frac{M^G}{S_G(M)}\xleftarrow{\iota,S_{G/H}}$$
$$\frac{M^H}{S(M)+\{\sum_{\rho \bmod H}(\rho-1)u_\rho\,|\,u_\rho\in M^H\}}\xleftarrow{\tau',1}$$
$$\frac{\{u\in M^H\,|\,S_{G/H}(u)=0\}}{\{\sum_{\rho \bmod H}(\rho-1)u_\rho\,|\,u_\rho\in M^H\}}\xleftarrow{\varphi',S_H}\frac{\{u\in M\,|\,S_G(u)=0\}}{\{\sum_{\sigma\in G}(\sigma-1)u_\sigma\,|\,u_\sigma\in M\}},$$

where inducing maps are indicated besides the maps themselves. The exactness at the 2nd and the 3rd terms is given in Artin-Tate [1]. The verification of the exactness at the 4th and 5th terms is also straightforward.

We turn to $(2_1')$. Though it is not difficult to achieve our aim also by direct verification, we had perhaps better to apply the method which has been utilized by Hattori [4] in proving I. Thus, with a G-module M, let $\tilde M$ be a free G-module of which M is a G-homomorphic image and N be the kernel of the homomorphism:

$$(*)\qquad 0\longrightarrow N\longrightarrow \tilde M\longrightarrow M\longrightarrow 0\quad\text{(exact).}$$

Let H be a subgroup of G. Then we have the exact sequence $0\longrightarrow N^H\longrightarrow\tilde M^H\longrightarrow M^H\longrightarrow H^1(H, N)\longrightarrow 0$. If H is an invariant subgroup, as we shall assume, then this is an exact sequence of G/H-modules. If $H^1(H, N)=0$, then

$$(**)\qquad 0\longrightarrow N^H\longrightarrow\tilde M^H\longrightarrow M^H\longrightarrow 0\quad\text{(exact).}$$

Provided that H is finite, the condition $H^1(H, N)=0$ is equivalent, because of $(*)$, to $H^0(H, M)=0$, which we shall assume together with the finiteness of G itself. Since $\tilde M^H$ is G/H-free, we have canonical isomorphisms $H^0(G/H, N^H)\cong H^{-1}(G/H, M^H)$, $H^{-1}(G/H, N^H)\cong H^{-2}(G/H, M^H)$. We have also canonical isomorphisms $H^0(G, N)\cong H^{-1}(G, M)$, $H^{-1}(G, N)\cong H^{-2}(G, M)$ and $H^0(H, N)\cong H^{-1}(H, M)$ (which are independent of the assumption $H^0(H,M)=0$). The last isomorphism commutes with the operation of G, and therefore, we have $H^0(H,N)_G\cong H^{-1}(H,M)_G$. Thus the terms of the sequence (2′) are isomorphic to the corresponding terms of the sequence which we obtain on replacing M in ($2'_0$) by N. This sequence is exact. So if we can show the commutativity of the diagram consisting of this sequence (i.e. ($2'_0$) with M replaced by N) and ($2'_1$) with corresponding terms connected by the arrows representing their canonical isomorphisms, then the exactness of ($2'_1$) will follow. But the verification of the commutativity is rather immediate. We shall consider the central square

$$(\natural)\qquad \begin{array}{ccc} H^0(H,N)_G & \xleftarrow{\tau'} & H^{-1}(G/H,N^H) \\ \uparrow\delta & & \uparrow\delta \\ H^{-1}(H,M)_G & \xleftarrow{\tau'} & H^{-2}(G/H,M^H) \end{array}$$

for an example. Let $\{\rho\}$ be a representative system of H-cosets in G, and let $\sum_\rho(\bar\rho)\otimes u(\rho)$ be a 1-cycle on G/H in M^H. Its class is mapped by δ to the class $\in H^{-1}(G/H,N^H)$ of the 0-cycle $\sum_\rho(\rho^{-1}-1)\tilde u(\rho)$ on G/H in N^H, where $\tilde u(\rho)$ is, for each ρ, an element of $\tilde M^H$ whose image in M^H is $u(\rho)$. This class is in turn mapped by τ' to the element of $H^0(H,N)_G$ represented by the class $\in H^0(H,N)$ of the 0-cocycle $\sum_\rho(\rho^{-1}-1)\tilde u(\rho)$ on H in N. On the other hand, the class of the 1-cycle $\sum_\rho(\bar\rho)\otimes u(\rho)$ is mapped by τ' to the element of $H^{-1}(H,M)_G$ represented by the class of the 0-cycle $\sum_\rho(\rho^{-1}-1)v(\rho)$ on H in M, where $S_H(v(\rho))=u(\rho)$. Let $\tilde v(\rho)$ be, for each ρ, an element of $\tilde M$ whose image in M is $v(\rho)$. Then our element of $H^{-1}(H,M)_G$ is mapped by δ to the element of $H^0(H,N)_G$ represented by the class of the 0-cocycle $S_H(\sum_\rho(\rho^{-1}-1)\tilde v(\rho))=\sum_\rho(\rho^{-1}-1)S_H(\tilde v(\rho))$ on H in N. Here $S_H(\tilde v(\rho))$ is mapped to $S_H(v(\rho))=u(\rho)$ by the homomorphism $\tilde M^H\longrightarrow M^H$. Hence, as the above element $\tilde u(\rho)$ we may take this element $S_H(\tilde v(\rho))$, for each ρ, having thus $\sum_\rho(\rho^{-1}-1)\tilde u(\rho)=\sum_\rho(\rho^{-1}-1)S_H(\tilde v(\rho))$. Then the commutativity of the above square is thus proved.

Duals to ($2'_0$), ($2'_1$) are ($1'_{-1}$), ($1'_0$) which are given also at the opening of the note. The maps $\lambda',\rho,\tau',\lambda'$ in ($1'_{-1}$) are induced, in the sense at hand, by the maps $1, S_{G/H}, 1, S_H$ respectively. As for ($1'_0$), which is valid and exact under the assumption $H^{-1}(H,M)=0$, it is perhaps sufficient to explain the map τ' in it. Thus, let u be an element of M^H such that for each $\rho\in G$ the element $(\rho-1)u$ lies in $S_H(M)$. As a 0-cocycle on H in M, it determines an element of $H^0(H,M)^G$ and this latter element is, by definition, mapped by τ' to the class of the 1-cocycle f on G/H in M_H such that $f(\bar\rho)$ is the unique counter image in M_H of $(\rho-1)u$ by S_H; observe that M_H is mapped into M monomorphically because of the assumption $H^{-1}(H,M)=0$. ($1'_1$), (1′) are two predecessors of a series of exact sequences (1′) derived from (1) under the additional assumption $H^{-1}(H, M)=H^0(H, M)=0$ by virtue of the then valid isomorphisms $M^H \cong M_H$.

Remark. In a sense it is more natural to define τ' in $(2'_0)$ or $(1'_{-1})$ to be the negative of the above defined map, which would make the diagram $(\natural)$, or the corresponding diagram for the transition from $(1'_{-1})$ to $(1'_0)$, anti-commutative.

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