163. *On Interpolations of Analytic Functions. II* (*Fundamental Results*)

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2. In this Note we consider a generalization of the result mentioned in the introduction of this paper.

Let D be a bounded closed points set whose complement K with respect to the extended plane is connected and regular in the sense that K possesses a Green's function with pole at infinity. Let $w=\phi(z)$ map K onto the region $|w|>1$ so that the points at infinity correspond to each other. Let Γ_ρ be the level curve determined by $|w|=\rho>1$.

Let the sequence of points (P) which lie on D satisfy the condition that the sequence of functions

$$\frac{W_n(z)}{\Delta^n w^n}=\frac{(z-z_1^{(n)})(z-z_2^{(n)})\cdots(z-z_n^{(n)})}{[\Delta\phi(z)]^n}$$

converges to a function $\lambda(w)$, single valued, analytic and non-vanishing for w exterior to the unit circle $|w|=1$, and uniformly on any bounded closed points set exterior to the unit circle, that is

$$\lim_{n\to\infty}\frac{W_n(z)}{[\Delta w]^n}=\lambda(w)\neq 0 \qquad \text{for} \qquad |w|>1, \tag{17}$$

where Δ is the capacity of D.

Let $f(z)$ be a function single valued and analytic throughout the interior of the level curve $\Gamma_\rho: |w|=|\phi(z)|=\rho>1$ but not analytic regular on Γ_ρ. Then the sequence of polynomials $P_n(z; f)$ of respective degrees n which interpolate to $f(z)$ in all the zeros of $W_{n+1}(z)$ is given by

$$P_n(z; f)=\frac{1}{2\pi i}\int_{\Gamma_R}\frac{W_{n+1}(t)-W_{n+1}(z)}{W_{n+1}(t)}\,\frac{f(t)}{t-z}dt: \qquad (1<R<\rho) \tag{18}$$

and we have, for z which satisfies $|\phi(z)|=|w|<R$,

$$R_n(z; f)\equiv f(z)-P_n(z; f)=\frac{1}{2\pi i}\int_{\Gamma_R}\frac{W_{n+1}(z)}{W_{n+1}(t)}\,\frac{f(t)}{t-z}dt: \quad (1<R<\rho). \tag{19}$$

In this case we have the following theorem.

Theorem 1. *Let D be a closed limited points set whose complement K with respect to the extended plane is connected and regular in the sense that K possesses a Green's function with pole at infinity. Let $W=\phi(z)$ map K onto the region $|w|>1$ so that the points at infinity correspond to each other.*

Let the function $f(z)$ be single valued and analytic throughout *the interior of the level curve* $\Gamma_\rho: |w|=|\phi(z)|=\rho>1$ *but not analytic regular on* Γ_ρ, *and* (P) *be a sequence of points sets which satisfies the condition (17).*

Then the sequence of polynomials $P_n(z;f)$ *of respective degrees* n *found by interpolation to* $f(z)$ *in all the zeros of* $W_{n+1}(z)$ *converges to* $f(z)$ *at every point interior to* Γ_ρ, *uniformly on any closed set interior to* Γ_ρ, *and diverges at every point exterior to* Γ_ρ. *Moreover, we have*

$$(20)\qquad \overline{\lim}_{n\to\infty}|R_n(z;f)|^{\frac{1}{n}}=\frac{|w|}{\rho}\qquad for\quad 1<|w|=|\phi(z)|<\rho,$$

and

$$(21)\qquad \overline{\lim}_{n\to\infty}|P_n(z;f)|^{\frac{1}{n}}=\frac{|w|}{\rho}\qquad for\quad 1<|w|=|\phi(z)|>\rho.$$

The first part of the theorem follows easily from the equations (20) and (21).

If we put $f(z)=F(w)$, the function $F(w)$ is single valued and analytic throughout the interior of the region between two circles $C_\rho: |w|=\rho>1$ and $C_r: |w|=r\leqq 1$, but not analytic regular on C_ρ by conditions of $f(z)$. Hence the function $F(w)$ can be expanded into Laurent's series

$$F(w)=\sum_{n=-\infty}^{\infty}A_n\left(\frac{w}{\rho}\right)^n=\sum_{n=-\infty}^{\infty}a_n\lambda_n\left(\frac{w}{\rho}\right)^n,$$

where a_n and λ_n satisfy (4), (5) and (6).

At first we prove the equation (20). From the equation (19), we have

$$R_n(z;f)=\frac{1}{2\pi i}\int_{1<|\zeta|=R<\rho}\frac{W_{n+1}(z)}{W_{n+1}(t)}\frac{F(\zeta)}{t-z}\frac{d\zeta}{\phi'(t)}:\ \zeta=\phi(t),\ 1<|w|<R,$$

where $\phi'(t)$ is non-vanishing on K. If we put for any point z lying on K and interior to the level curve Γ_ρ

$$\begin{aligned}\varphi_n(\zeta)\equiv\varphi_n(\zeta;z)&=\frac{W_{n+1}(z)}{W_{n+1}(t)}\left(\frac{\zeta}{w}\right)^{n+1}\frac{1}{(t-z)\phi'(t)}\\&=\frac{(\Delta\zeta)^{n+1}}{W_{n+1}(t)}\frac{W_{n+1}(z)}{(\Delta w)^{n+1}}\frac{1}{(t-z)\phi'(t)}:\ \zeta=\phi(t),\ w=\phi(z),\end{aligned}$$

the sequence of functions $\varphi_n(\zeta)$ converges uniformly to $\dfrac{\lambda(w)}{\lambda(\zeta)}\dfrac{1}{(t-z)\phi'(t)}$ single valued, analytic and non-vanishing on a closed domain $1<|w|<R''\leqq|\zeta|\leqq R'$, $(R''<\rho<R')$.

By Lemma 3, we can verify that

$$\begin{aligned}&\overline{\lim}_{n\to\infty}\frac{1}{\lambda_n}\left|\left(\frac{\rho}{w}\right)^{n+1}R_n(z;f)\right|\\&\qquad=\overline{\lim}_{n\to\infty}\frac{\rho}{\lambda_n}\left|\frac{\rho^n}{2\pi i}\int_{C_R}\varphi_n(\zeta)F(\zeta)\zeta^{-n-1}d\zeta\right|:1<|w|<R<\rho\end{aligned}$$

$$=\overline{\lim}_{n\to\infty}\frac{\rho}{\lambda_n}|\gamma_n^{(n)}| \qquad \text{for} \quad 1<|w|=|\phi(z)|<\rho$$

is bounded and positive, where $\gamma_k^{(n)}$ are defined by

$$\varphi_n(\zeta)F(\zeta)=\sum_{k=-\infty}^{\infty}\gamma_k^{(n)}\left(\frac{\zeta}{\rho}\right)^k.$$

Accordingly we can verify the equation

$$\lim_{n\to\infty}|R_n(z;f)|^{\frac{1}{n}}=\frac{|w|}{\rho} \qquad \text{for} \quad \rho>|w|=|\phi(z)|>1.$$

Thus the sequence of polynomials $P_n(z;f)$ converges to $f(z)$ on K interior to Γ_ρ, consequently at every point interior to Γ_ρ, and converges uniformly on any closed set interior to Γ_ρ. Thus the convergence of $P_n(z;f)$ has been proved.

Next we shall prove the relation (21). From the equation (18), we have

$$P_n(z;f)=\frac{1}{2\pi i}\int_{1<|\zeta|=R<\rho}\frac{W_{n+1}(t)-W_{n+1}(z)}{W_{n+1}(t)}\frac{F(\zeta)}{t-z}\frac{d\zeta}{\phi'(t)};\ \zeta=\phi(t),\ w=\phi(z).$$

For any point z exterior to the level curve Γ_ρ, the sequence of functions

$$\varphi_n(\zeta)\equiv\varphi_n(\zeta;z)=\frac{W_{n+1}(t)-W_{n+1}(z)}{W_{n+1}(t)(t-z)}\left(\frac{\zeta}{w}\right)^{n+1}\frac{1}{\phi'(t)}$$
$$=\frac{(\Delta\zeta)^{n+1}}{W_{n+1}(t)}\left\{\frac{W_{n+1}(t)}{(\Delta w)^{n+1}}-\frac{W_{n+1}(z)}{(\Delta w)^{n+1}}\right\}\frac{1}{(t-z)\phi'(t)}$$

converges uniformly to $\dfrac{-\lambda(w)}{\lambda(\zeta)(t-z)\phi'(t)}$ single valued, analytic and non-vanishing on a closed domain $1<R''\leqq|\zeta|\leqq R'<|w|$, $(R''<\rho<R')$. By Lemma 3 we can verify that

$$\overline{\lim}_{n\to\infty}\frac{1}{\lambda_n}\left|\left(\frac{\rho}{w}\right)^{n+1}P_n(z;f)\right|$$
$$=\overline{\lim}_{n\to\infty}\frac{\rho}{\lambda_n}\left|\frac{\rho^n}{2\pi i}\int_{|\zeta|=R}\varphi_n(\zeta)F(\zeta)\zeta^{-n-1}d\zeta\right|$$
$$=\overline{\lim}_{n\to\infty}\frac{\rho}{\lambda_n}|\gamma_n^{(n)}| \qquad \text{for } z \text{ exterior to } \Gamma_\rho,$$

is bounded and positive. Thus the relation

$$\overline{\lim}_{n\to\infty}|P_n(z;f)|^{\frac{1}{n}}=\frac{|w|}{\rho} \qquad \text{for } z \text{ exterior to } \Gamma_\rho(|\phi(z)|>\rho)$$

follows at once. Accordingly, the sequence of polynomials $P_n(z;f)$ of respective degrees n diverges at every point exterior to the level curve Γ_ρ. Thus the theorem has been established.

3. In this paragraph, we consider some examples of the theorem fore-mentioned.

Let $\mu(\theta)\geqq 0$ be a function defined and measurable in the interval $-\pi\leqq\theta\leqq\pi$, for which the integrals

$$(22) \qquad \int_{-\pi}^{\pi} \mu(\theta) d\theta > 0, \qquad \int_{-\pi}^{\pi} |\log \mu(\theta)| \, d\theta$$

exist. With such a function $\mu(\theta)$ we can associate a uniquely determined analytic function $D(z; \mu) \equiv D(z)$, regular and non-zero for $|z|<1$ with $D(0)>0$, which satisfies

$$(23) \qquad \lim_{r\to 1-0} D(re^{i\theta}) = D(e^{i\theta}), \qquad \mu(\theta) = |D(e^{i\theta})|^2.$$

Let $\{\phi_n(z)\}$ be the set of ortho-normal polynomials on the unit circle $C: |z|=1$ corresponding to the weight function $\mu(\theta)$, that is, $\phi_n(z)$ satisfy

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_n(z)\overline{\phi_m(z)}\, \mu(\theta) d\theta = \begin{cases} 1 : n = m \\ 0 : n \neq m, \end{cases} \quad \text{where} \quad z = e^{i\theta}.$$

Then it is known that, in the exterior of the unit circle, the sequence of function $\dfrac{\phi_n(z)}{z^n} = \dfrac{k_n z^n + k_{n-1} z^{n-1} + \cdots + k_0}{z^n}$ converges to the function $\{\overline{D}(z^{-1})\}^{-1}$ uniformly on any closed set exterior to the unit circle, that is, we have

$$\lim_{n\to\infty} \frac{\phi_n(z)}{z^n} = \lim_{n\to\infty} \frac{k_n z^n + k_{n-1} z^{n-1} + \cdots + k_0}{z^n} = \{\overline{D}(z^{-1})\}^{-1} : |z|>1.$$

It is easily verified that the first coefficient k_n of $\phi_n(z)$ satisfies

$$\lim_{n\to\infty} k_n = \lim_{n\to\infty} \lim_{z\to\infty} z^{-n} \phi_n(z) = \{D(0)\}^{-1}.$$

Thus we have

$$\lim_{n\to\infty} \frac{k_n^{-1}\phi_n(z)}{z^n} = \lim_{n\to\infty} \frac{z^n + \cdots}{z^n} = D(0)\{\overline{D}(z^{-1})\}^{-1} : |z|>1,$$

where the function defined by the last term is single valued, analytic and non-vanishing in the exterior of the unit circle.

Accordingly, we can verify that the sequence of points

$$z_1^{(n)}, z_2^{(n)}, \cdots, z_n^{(n)}; \qquad n = 1, 2, \cdots$$

defined by all the zeros of $\phi_n(z)$ satisfies the condition of Theorem 1. Now a theorem follows at once as a corollary of Theorem 1.

Theorem 2. *Let $f(z)$ be a function single valued and analytic throughout the interior of the circle $C_\rho: |z|=\rho>1$. Let $\{\Phi_n(z)\}$ be the ortho-normal set of polynomials $\phi_n(z)$ of respective degrees n corresponding to a weight function $\mu(\theta): -\pi \leqq \theta \leqq \pi$ for which the integrals (22) exist. Then the sequence of polynomials $P_n(z; f)$ of respective degrees n found by interpolation to $f(z)$ in all the zeros of $\phi_{n+1}(z)$ converges to $f(z)$ at every point interior to C_ρ, uniformly on any closed points set interior to C_ρ. And $\{P_n(z; f)\}$ diverges at every point exterior to C_ρ as n tends to infinity.*

The first part of this theorem (the convergence of the sequence $P_n(z; f)$) has been found by Szegö[1] in the more generalized form.

1) G. Szegö: Über orthogonale Polynome, Math. Z., **9**, 218–270 (1921).

Let $S_n(z; f)$ be the partial sums of respective degrees n obtained by Fourier-expansion of $f(z)$ by the ortho-normal set $\{\phi_n(z)\}$. Then it is known that the sequence of polynomials $S_n(z; f)$ converges to $f(z)$ at every point interior to C_ρ and diverges at every point exterior to C_ρ. Accordingly, we can verify that the exact convergence-region of the sequences $\{P_n(z; f)\}$ and $\{S_n(z; f)\}$ coincide to each other.

Theorem 2 can be generalized to such a case that the sequence of points (P) is determined by the zeros of an ortho-normal set defined on a more general curve on z-plane. But we shall consider only the case such that an ortho-normal set is defined on the real segment $[-1, 1]$.

Let $w(x) \geqq 0$ be a weight function on the interval $-1 \leqq x \leqq 1$ such that for $\mu(\theta) = w(\cos\theta)|\sin\theta|$ the integrals (22) exist. If $D(w; \mu) \equiv D(w)$ denotes the analytic function corresponding to $\mu(\theta)$ in the sense fore-mentioned, it is known[2] that the set of ortho-normal polynomials $p_n(x)$ of respective degrees n, associated with the weight function $w(x)$, satisfies

$$\lim_{n\to\infty} w^{-u} p_n(z) = \lim_{n\to\infty} \frac{k_n z^n + k_{n-1} z^{n-1} + \cdots + k_0}{w^n}$$

$$= (2\pi)^{-\frac{1}{2}} \{D(w^{-1})\}^{-1} \qquad \text{for } |w| > 1,$$

where z is in the complex plane cut along the real segment $[-1, 1]$ and $z = \frac{1}{2}(w + w^{-1})$.

The first coefficient k_n of $p_n(z)$ satisfies

$$\lim_{n\to\infty} 2^{-n} k_n = \lim_{n\to\infty} \lim_{z\to\infty} (2z)^{-n} p_n(z)$$

$$= \lim_{n\to\infty} \lim_{w\to\infty} w^{-n} p_n(z) = (2\pi)^{-\frac{1}{2}} \{D(0)\}^{-1},$$

and the capacity of the real segment $[-1, 1]$ is known to be equal to $\frac{1}{2}$. Now the equation

$$\lim_{n\to\infty} \frac{k_n^{-1} p_n(z)}{2^{-n} w^n} = \lim_{n\to\infty} \frac{z^n + \cdots}{2^{-n} w^n} = (2\pi)^{-\frac{1}{2}} D(0) \{D(z^{-1})\}^{-1}$$

follows at once. Accordingly, we can verify that the sequence of points

$$x_1^{(n)}, x_2^{(n)}, \cdots, x_n^{(n)}; \qquad n = 1, 2, \cdots$$

defined by all the zeros of $p_n(z)$ satisfies the condition of Theorem 1.

In this case, the level curve $\Gamma_\rho : |w| = \rho > 1$ is defined by the ellipse with foci at ± 1 and with semi-axes $\frac{1}{2}(\rho + \rho^{-1})$ and $\frac{1}{2}(\rho - \rho^{-1})$. Thus we have

Theorem 3. *Let $f(z)$ be a function single valued and analytic throughout the interior of the ellipse $\Gamma_\rho : \rho > 1$ with foci at ± 1 and*

2) G. Szegö: Orthogonal polynomials, Am. Math. Soc. Coll. Publ., **23** (1939).

with semi-axes $\frac{1}{2}(\rho+\rho^{-1})$ *and* $\frac{1}{2}(\rho-\rho^{-1})$ *but not analytic regular on* Γ_ρ. *Let* $\{p_n(x)\}$ *be the set of ortho-normal polynomials* $p_n(x)$ *of respective degrees* n *corresponding to a weight function* $w(x)$ *on* $1\leqq x\leqq -1$ *which satisfies the conditions fore-mentioned.*

Then the sequence of polynomials $P_n(z;f)$ *of respective degrees* n *found by interpolation to* $f(z)$ *in all the zeros of* $p_n(x)$ *converges to* $f(z)$ *at every point interior to* Γ_ρ, *uniformly on any closed set interior to* Γ_ρ. *And the sequence* $\{P_n(z;f)\}$ *diverges at every point exterior to* Γ_ρ *as* n *tends to infinity.*