8. *The Initial Value Problem for Linear Partial Differential Equations with Variable Coefficients. I*

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Recently Yosida[1] gives the existence theorem for Cauchy problem of the wave equation by semigroup-like method, but it seems me not so adequate to obtain solutions for more general equations such that their coefficients depend on the time.

The object of this paper is to give an existence theorem for Cauchy problem in the whole space, and gives some generalization of Leray's results.[2] Our main idea owes to Yosida.

1. Notations. We consider only a real function or a smooth system of real linear differential operator defined over l-Euclidean vector space R^l_x and $l+1$-Euclidean vector space $R^1_t\times R^l_x$: that is, their coefficients have bounded derivatives of all orders.

Let $B\left(x, \frac{\partial}{\partial x}\right)=\left(b_{ij}\left(x, \frac{\partial}{\partial x}\right)\right)$ $(i, j=1,2,\cdots, m)$ be an (m, m)-smooth system of differential operator defined over R^l_x. Let $s=\{s(i)\,|\,i=1,2, \cdots, m\}$ be the set of non negative integers such that the order $o(b_{ij})$ of b_{ij} does not exceed $s(i)+s(j)$ and denote by $b'_{ij}\left(x, \frac{\partial}{\partial x}\right)$ the sum of terms in b_{ij} with order exactly $s(i)+s(j)$. Then we call that $B\left(x, \frac{\partial}{\partial x}\right)$ is *uniformly strongly elliptic*, if there is a positive ρ such that

$$b'_{ij}(x, i\xi)\, v_i\bar{v}_j \geqq \rho\sum_{i=1}^{m}|\xi|^{2s(i)}\,|v_i|^2$$

for all real scalars $\xi:(\xi_1,\cdots, \xi_l)$, all $x\in R^l_x$ and all complexes $v_1,\cdots, v_m$. By the smoothness of $B\left(x, \frac{\partial}{\partial x}\right)$ it is equivalent to the following: $(b'_{ij}(x, i\xi))$ is positive definite for all $x\in R^l_x$ and all real scalars $\xi:|\xi|=1$. Then from Leray's method, it is shown that there are positive k, α and β such that

$$((u, v))_B=\int_{R^l_x}\left(\left(B\left(x, \frac{\partial}{\partial x}\right)+B\left(x, \frac{\partial}{\partial x}\right)^* +k\right)u, v\right)dx,$$

1) Cf. K. Yosida: An operator-theoretical integration of the temporally inhomogeneous wave equation (to appear).

2) Cf. L. Leray: Hyperbolic equations with variable coefficients, Princeton, N. J. (1954).

$$((u, v))_s=\sum_{i=1}^{m}\int_{R_x^l} Q^{s(i)}\, u_i v_i\, dx,$$

$$\alpha((u, v))_s \geqq ((u, v))_B \geqq \beta((u, v))_s,$$

where $u=(u_1,\cdots, u_m)\in \mathfrak{D}_x^{(m)}$, i.e., u_i $(i=1, 2,\cdots, m)$ are infinitely differentiable functions with compact carries defined on R_x^l, and

$$Q=1-\sum_{i=1}^{l}\frac{\partial^2}{\partial x_i^2}.$$

Furthermore let $B\left(t, x, \frac{\partial}{\partial x}\right)$ be a smooth system defined over $R_t^1\times R_x^l$, which does not contain $\frac{\partial}{\partial t}$ and which is uniformly strongly elliptic with the same s for all $t\in R_t^1$. This operator is simply denoted by B_t (when $t\in R_t^1$ is fixed), $B^{(s)}$ or B, and we introduce Hilbert spaces which are isomorphic:

H_{B_t}, the completion of functions in $\mathfrak{D}_x^{(m)}$ under the inner product $((u, v))_{B_t}$.

H_s, the completion of functions in $\mathfrak{D}_x^{(m)}$ under the inner product $((u, v))_s$.

Finally by $\mathfrak{E}_t(\mathfrak{D}_{L^2(x)}^{(m)})$ we denote the space of infinitely differentiable vector valued functions defined on R_t^1 into $\mathfrak{D}_{L^2(x)}^{(m)}$, where $\mathfrak{D}_{L^2(x)}^{(m)}$ is the projective limit of H_s's.

2. Theorems. Let $A\left(t, x, \frac{\partial}{\partial x}\right)$ be an (m, m)-smooth system defined over $R_t^1\times R_x^l$ and let $B\left(t, x, \frac{\partial}{\partial x}\right)$ be an other (m, m)-smooth system defined in Section 1 where the associated $s(i)$ $(i=1,2,\cdots, m)$ are sufficiently large.

We say that A is *semi-bounded by the norm defined by B*, if there is a positive continuous function γ_t defined over R_t^1 such that

$$\int_{R_x^l}\left(A\left(t, x, \frac{\partial}{\partial x}\right)u(x),\ B\left(t, x \frac{\partial}{\partial x}\right)u(x)\right)dx$$

$$\leqq \gamma_t\int_{R_x^l}\left(B\left(t, x, \frac{\partial}{\partial x}\right)u(x), u(x)\right)dx$$

for any $t\in R_t^1$ and any $u\in \mathfrak{D}_x^{(m)}$

Then the above inequality implies that

(1) $$((A_t u, u))_{B_t}\leqq \gamma_t((u, u))_{B_t},$$

since $s(i)$ is sufficiently large.

From (1) we see the following

Theorem 1. *If $A\left(t, x, \frac{\partial}{\partial x}\right)$ is semi-bounded by the norm defined by $B^{(s)}$ and by $B^{(s')}$ such that $s(i)$ is sufficiently larger than $s'(i)$*

$(i=1,2,\cdots,m)$. *Then the weak extension* $\bar{A}_t$ *of* A_t *in* H_{B_t} *into itself satisfies the following conditions: there exist positive real* λ_0 *and* β_t *such that*

a) *the real* $\lambda \geqq \lambda_0$ *belongs to the resolvent set of* $\bar{A}_t$,

b) $$\left\| \left(I-\frac{1}{\lambda}\bar{A}_t\right)^{-1} \right\|_{B_t^s} \leqq 1+\frac{\beta_t}{\lambda}$$

From Theorem 1 and by a modification of Yosida's method we see the following

Theorem 2. *If* $A\left(t, x, \frac{\partial}{\partial x}\right)$ *is semi-bounded by the norm defined by* $B^{(s)}$*'s and* $s(i)\underset{s}{\uparrow}\infty$ *for* $i=1,2,\cdots,m$. *Then Cauchy problem for* $\frac{\partial}{\partial t}I-A\left(t, x, \frac{\partial}{\partial x}\right)$ *is well posed in the following sense: For any* $v\in\mathfrak{E}_t(\mathfrak{D}^{(m)}_{L^2(x)})$ *and any* $g\in\mathfrak{D}^{(m)}_{L^2(x)}$, *there is uniquely a* $u\in\mathfrak{E}_t(\mathfrak{D}^{(m)}_{L^2(x)})$ *such that*

$$\left(\frac{\partial}{\partial t}-A\left(t, x, \frac{\partial}{\partial x}\right)\right)u=v \qquad \text{for } t\geqq 0$$

$$u(0, x)=g(x).$$

3. Outline of proofs of Theorems. The inequality (1) implies that

$$(2) \qquad \left\| \bar{B}_t^{-1}\left(1-\frac{1}{\lambda}A_t^*\right)B_t u \right\|_{B_t} \geqq \left(1+\frac{2\gamma_t}{\lambda}\right)^{-1} \| u \|_{B_t},$$

$$(3) \qquad \left\| \left(1-\frac{1}{\lambda}A_t\right)u \right\|_{B_t} \geqq \left(1+\frac{2\gamma_t}{\lambda}\right)^{-1} \| u \|_{B_t}$$

for any $u\in\mathfrak{D}^{(m)}_{(x)}$ and for any $\lambda>2\gamma_t$. Here replacing B_t by $B_t+B_t^*+k$ we denote by $\bar{B}_t$ the weak extension of B_t. Then (2) implies the existence of the weak solution of $1-\frac{1}{\lambda}A_t$ and from (3) where $B_t=B_t^{(s')}$, it follows the uniqueness of such weak solution. Thus we see Theorem 1. Here we remark that in (b) we can not in general replace $\| \quad \|_{B_t^{(s)}}$ by $\| \quad \|_s$

Furthermore under the assumption of Theorem 2, from Theorem 1 it follows that (a) and (b) hold for $B^{(s)}$ with sufficiently large $s(i)$. Then from (b) and by a limit-process we see that

(c) $\left(1-\frac{1}{n}A_t\right)^{-1}$ *is strongly continuous with resp. to* t *in* H_s, *for fixed* $\lambda\geqq\lambda_0$, *where* λ_0 *depends on* s.

From (a), (b) and using a certain estimate with respect to $B_t^{(s)}$ (c) a suitable modification of Yosida's method is applicable to obtain our Theorem 2. Finally considering the iteration $\frac{\partial^i u}{\partial t^i}$ it follows the differentiability (in the strong sense), of the solution, with respect to t.

4. Examples. 1. *Parabolic equations.* Let $A\left(t, x, \frac{\partial}{\partial x}\right)$ be a smooth system such that for fixed t it is uniformly strongly elliptic. Then Cauchy problem for $\frac{\partial}{\partial t}+A\left(t, x, \frac{\partial}{\partial x}\right)$ is well posed in the sense of Theorem 2.

For let $B\left(t, x, \frac{\partial}{\partial x}\right)$ be $Q^s I$, where s is a positive integer. Then since $Q^s A_t$ is also uniformly strongly elliptic, there is a continuous positive function γ_t such that

$$(Q^s I)(-A_t)+(-A_t^*)(Q^s I)\leqq \gamma_t Q^s I,$$

where $B\geqq C$ means that

$$\int_{R_x^l}(Bu, u)dx\geqq \int_{R_x^l}(Cu, u)dx \qquad \text{for } u\in \mathfrak{D}_x^{(m)}$$

Thus Theorem 2 is applicable to $\frac{\partial}{\partial t}-A\left(t, x, \frac{\partial}{\partial x}\right)$

2. *Hyperbolic equations.* If a smooth operator $\mathbf{a}\left(t, x, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)$ of order m:

$$\frac{\partial^m}{\partial t^m}+\alpha_0\left(t, x, \frac{\partial}{\partial x}\right)\frac{\partial^{m-1}}{\partial t^{m-1}}+\cdots+\alpha_{m-1}\left(t, x, \frac{\partial}{\partial x}\right)$$

satisfies the following conditions: let α_i' be the principal part of α_i and let

$$a'(t, x, \xi)=\xi_0^m+\alpha_0'(t, x, \xi^*)\xi_0^{m-1}+\cdots+\alpha_{m-1}'(t, x, \xi^*),$$

where $\xi^*=(0, \xi_1, \xi_2,\cdots, \xi_l)$. Then the equation of λ

$$a'(t, x, \lambda(\xi_0, 0,\cdots, 0)+(0, \eta_1, \eta_2,\cdots, \eta_l))=0$$

has real roots $U_i(x, t, \eta^*)$ such that

$$|U_i(x, t, \eta^*)-U_j(x, t, \eta^*)|\geqq b$$

for some positive b, for different i, j $(i, j=1, 2,\cdots, m)$, any $(x, t)\in R_t^1\times R_x^l$ and any real scalar $\eta^*:|\eta^*|=1$.

Then by Leray it is shown that Theorem 2 is applicable to solve Cauchy problem for $\mathbf{a}$ with the initial hyperspace: $t=0$. Here we remark that, without reducing to a system, the unique existence of solution of Cauchy problem for $\mathbf{a}$ follows directly from the duality of Hilbert spaces.

3. Let $A\left(t, x, \frac{\partial}{\partial x}\right)=\begin{pmatrix} 0 & ,1 \\ -\mathbf{a}\left(t, x, \frac{\partial}{\partial x}\right) & ,0\end{pmatrix}$, where $\mathbf{a}\left(t, x, \frac{\partial}{\partial x}\right)$ is a uniformly strongly elliptic operator for all $t\in R_t^1$ and the order of $\mathbf{a}$ is $2m$. Then for any positive integer s, $\mathbf{a}_t^s$ is also uniformly strongly elliptic, i.e. there are $k_t^{(s)}$ and $k_t^{(s-1)}$ such that

$$\mathbf{a}_t^{s-1}+\mathbf{a}_t^{*s-1}+k_t^{(s-1)}\geqq \alpha_t Q^{m(s-1)}$$

$$\mathbf{a}_t^s+\mathbf{a}_t^{*s}+k_t^{(s)}\geqq \alpha_t Q^{ms} \qquad \text{for some positive } \alpha_t.$$

Let $B\left(t, x, \frac{\partial}{\partial x}\right)=\begin{pmatrix} \mathbf{a}_t^s+\mathbf{a}_t^{*s}+k_t^{(s)}, & 0 \\ 0 & , \mathbf{a}_t^{s-1}+\mathbf{a}_t^{*s-1}+k_t^{(s-1)} \end{pmatrix}$

Then if $o(\mathbf{a}_t-\mathbf{a}_t^*)\leqq m$,

$$B_tA_t+A_t^*B_t=\begin{pmatrix} 0 & , (\mathbf{a}_t-\mathbf{a}_t^*)\mathbf{a}_t^{s-1}-k_t^{(s-1)}\mathbf{a}_t^*+k_t^{(s)} \\ \mathbf{a}_t^{*s-1}(\mathbf{a}_t^*-\mathbf{a}_t)-k_t^{(s-1)}\mathbf{a}_t+k_t^{(s)}, & 0 \end{pmatrix}$$

Since $o((\mathbf{a}_t-\mathbf{a}_t^*)\mathbf{a}_t^{s-1}-k_t^{(s-1)}\mathbf{a}_t^*+\mathbf{k}_t^{(s)})\leqq 2m(s-1)+m$, from Leray's lemmas we see that

$B_tA_t+A_t^*B_t\leqq\gamma_tB_t$ for a positive continuous function γ_t. Thereby Theorem 2 is applicable to the equation $\frac{\partial^2}{\partial t^2}+\mathbf{a}\left(t, x, \frac{\partial}{\partial x}\right)$ where $o(\mathbf{a}_t-\mathbf{a}_t^*)\leqq m$.

4. *Schrödinger equations.* Let $A\left(t, x, \frac{\partial}{\partial x}\right)=\begin{pmatrix} 0, & -\mathbf{a}\left(t, x, \frac{\partial}{\partial x}\right) \\ \mathbf{a}\left(t, x, \frac{\partial}{\partial x}\right), & 0 \end{pmatrix}$

where $\mathbf{a}\left(t, x, \frac{\partial}{\partial x}\right)$ is uniformly strongly elliptic and $\mathbf{a}=\mathbf{a}^*$ for any $t\in R_t^1$. For any positive integer s and for sufficiently large continuous function $k_t^{(s)}$

$$B\left(t, x, \frac{\partial}{\partial x}\right)=\begin{pmatrix} \mathbf{a}_t^s+k^{(s)}, & 0 \\ 0 & , \mathbf{a}_t^s+k^{(s)} \end{pmatrix}$$

Then for any positive γ

$$B_tA_t+A_t^*B_t=0<\gamma B_t.$$

Therefore Theorem 2 is applicable to the equation

$$\frac{\partial}{\partial t}\pm i\ \mathbf{a}\left(t, x, \frac{\partial}{\partial x}\right) \text{ where } \mathbf{a}_t=\mathbf{a}_t^*$$

5. Let $\mathbf{a}\left(\frac{\partial}{\partial x}\right)$ be a partial differential operator with constant coefficient defined over R^{l+1} such that

$$\mathbf{a}(\lambda\xi_0+i\eta)\neq 0$$

for $\xi_0=(a, 0,\cdots, 0)$ $a>0$, any η, and any real $\lambda\geqq 1-\varepsilon$ $(1>\varepsilon>0),\cdots$, where ξ_0, η are $(l+1)$-real vectors. Here we assume that the term in a (ξ) of the highest order with respect to the first variable is independent of other l variables. Then by the same method as in the earlier paper[3] it is shown that the carrier of the unique elementary solution $\mathbf{a}\left(\frac{\partial}{\partial x}\right)^{-1}\in\mathfrak{S}'(\xi_0)$ is contained in the half-space $(x, \xi_0)\geqq 0$. Hence Cauchy problem for $\mathbf{a}\left(\frac{\partial}{\partial x}\right)$ with the initial hyperspace $(x, \xi_0)=0$ is well posed in various senses.

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3) Cf. T. Shirota: On solutions of a partially differential equation with a parameter, Proc. Japan Acad., **32**, 401 (1956).

A generalization for the case of variable coefficient, in this direction, as well as the full proofs of the above results will be published in the Osaka Mathematical Journal.

Added in proof: While in proof reading I had access to a paper of S. Mizohata: Le problème de Cauchy pour les équations paraboliques, J. Math. Soc. Japan, 8 (1956). From his algebraic results (1.22–1.24) his fundamental Proposition 4 follows immediately from ours.