# 20. *Analytic Functions in the Neighbourhood of the Ideal Boundary*

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Let $R$ be a Riemann surface with null-boundary and let $\{R_n\}$ be its exhaustion with compact relative boundary. We proved the following

Theorem 1.[1] *Let $R'$ be a subsurface of $R$ with compact relative boundary. Let $f(z)$ be a bounded analytic function on $R'$. Then $f(z)$ has a limit as $z$ tends to an ideal boundary component of $R'$.*

We extend this theorem to more general class of Riemann surfaces. Let $R$ be a Riemann surface with positive boundary and let $R'$ be a subsurface of $R$ with compact relative boundary $\Gamma$. We introduce two classes of Riemann surfaces.

There exists no non-constant one valued bounded (Dirichlet bounded) harmonic function $U(z)$ on $R'$ such that $U(z)=0$ on $\Gamma$, the period of the conjugate function of $U(z)$ vanishes along every dividing cut of $R$. We say $R\in O'_{AB}$ and $\in O'_{AD}$ respectively. $O'_{AB}$ and $O'_{AD}$ are the extension of the classes of $O_{AB}$ and $O_{AD}$ of the Riemann surface of finite genus. We see easily that the property $\in O'_{AB}$ ($\in O'_{AD}$) is the one depending only on the ideal boundary.

Theorem 2. *Suppose a bounded (Dirichlet bounded) analytic function on $R'\in O'_{AB}(O'_{AD})$. Then $f(z)$ has a limit as $z$ tends to a boundary component of $R'$.*

To prove Theorem 2 we make some preparations.

Let $R$ be a Riemann surface with positive boundary and let $\{R_n\}$ $(n=0,1,2,\cdots)$ be its exhaustion with compact relative boundary $\{\partial R_n\}$. Let $N(z,p): p\in R$ be a positive harmonic function in $R-R_0$ such that $N(z,p)=0$ on $\partial R_0$, $N(z,p)$ has a logarithmic singularity at $p$ and $N(z,p)$ has the minimal $*$-Dirichlet integral.[2] Let $\{p_i\}$ be a sequence tending to the ideal boundary of $R$ such that $\{N(z,p_i)\}$ converges uniformly in every compact domain of $R$. We say that $\{p_i\}$ is a fundamental sequence determining an ideal boundary point and we make $\lim_{i=\infty} N(z,p_i)$ correspond to this ideal boundary point. Denote by $B$ the ideal boundary point. The distance between points $p_1$ and $p_2$ of $R-R_0+B$ is defined by

1) Z. Kuramochi: Potential theory and its applications, I, Osaka Math., **3** (1951).

2) Z. Kuramochi: Mass distributions on the ideal boundaries of abstract Riemann surfaces, II, Osaka Math., **8** (1956).

$$\delta(p_1, p_2) = \sup_{z \in R_1 - R_0} \left| \frac{N(z, p_1)}{1+N(z, p_1)} - \frac{N(z, p_2)}{1+N(z, p_2)} \right|.$$

Then $R-R_1+B$ and $B$ are closed and compact. We defined in the previous paper[3] *minimal* point, *singular minimal* point. Then we have the following

Lemma 1. *Let $p$ be a singular minimal point and let $v(p)$ be a neighbourhood with respect to $\delta$-metric. Then there exists no Dirichlet bounded analytic functions on $v(p)$.*

Lemma 2. *Suppose $R' \in O'_{AD}$. Then $R'$ has no boundary component of positive capacity.*[4]

In fact, assume that $p$ is a boundary component of positive capacity. Then we can construct easily a harmonic function $U(z)$ such that $U(z)=0$ on $\Gamma$, $D(U(z))<\infty$ and the conjugate of $U(z)$ has no period along every dividing cut.

Lemma 3. *Let $U_n(z)$ be a harmonic function on $R'$ such that $U_n(z)=$Real part of $f(z)$ on $\Gamma$ and $\frac{\partial U_n(z)}{\partial n}=0$ on $\partial R_n$. Then $U_n(z)$ converges to a harmonic function $U^*(z)$ in mean and moreover the conjugate of $U(z)$ has no period along every dividing cut, whence $U(z)\equiv Re\, f(z)$. We say such $U(z)$ a $*$-harmonic function. Then we have*

Lemma 4. *Every $*$-harmonic function satisfies the maximum and minimum principle.*

We denote by $\underline{B}$ the all ideal boundary components of $R$. We compactify $R$ by adding $\underline{B}$ to $R$ and introduce usually a topology on $R+\underline{B}$. Then $R+\underline{B}$ and $\underline{B}$ are closed and compact. We call this topology $A$-topology.

Lemma 5. *Let $F$ be a closed subset of $R'+B$ of capacity zero with respect to $A$-topology. Then there exists a positive harmonic function $V(z)$ on $R'$ such that $V(z)=0$ on $\Gamma$, $V(z)\to\infty$ as $z$ tends to $F$, $V(z)<\infty$ as $z$ tends to a point $\notin F$ and $V_M(z)=V(z)$, where $V_M(z)$ is a harmonic function such that $V_M(z)=0$ on $\Gamma$, $V_M(z)=M$ on $C_M=E[z\in R: V(z)=M]$ and has the minimal Dirichlet integral on the domain bounded by $\Gamma$ and $C_M$. Hence $\int_{C_M} \frac{\partial V(z)}{\partial n} ds \leqq \int_{\Gamma} \frac{\partial V(z)}{\partial n} ds$ for every $0<M<\infty$ and $\int_{C_M} \frac{\partial V(z)}{\partial n} ds = \int_{\Gamma} \frac{\partial V(z)}{\partial n} ds$ for every $M\notin E$ such that mes $E=0$.*

Remark. In the previous paper[5] we proved Lemma 5 under the condition that $F$ is closed in $\delta$-metric and $F\in R+B_1$, where $B_1$ is the set of minimal points. In this case, the above conditions are not

3)-5) See 2).

necessary.

Proof of Theorem 2. Suppose $R\in O'_{AD}$ and $D(f(z))<\infty$. We denote by $G_i$ the domain containing a subset of $\underline{B}$ and bounded by compact or non compact curves $\gamma_i$. Denote by $f(\gamma_i)$ the image of $\gamma_i$ by $f(z)$. Then by Lemma 4 $G_i\subset G_j$ implies that $f(G_i)$ is contained in $f(G_j)$. Let $p$ be a boundary component of $R'$. Apply Lemma 5 to $p$. Then $D(f(z))<\infty$ implies the existence of a sequence of curves $\{\gamma_i\}$ such that $f(G_i)$ is contained in $f(\gamma_i)$ and the length of $f(\gamma_i)$ tends to zero as $i$ tends to $\infty$. Hence $f(z)$ tends to a point $\prod_{i=1}^{\infty}\overline{f(G_i)}$.[6]

Next suppose $R\in O'_{AB}$ and $|f(z)|<M$. We can suppose without loss of generality that $f(z)$ is analytic on $\Gamma$. Consider $U(z)=Re\, f(z)$ and $U(z)$ has the minimal Dirichlet integral. Then $U(z)\equiv Re\, f(z)$ and $D(U(z))=\int_{\Gamma} U(z)\frac{\partial U(z)}{\partial n}\,ds$, which implies $D(f(z))<\infty$. On the other hand, we can easily prove $O'_{AB}\subset O'_{AD}$ by the same method to prove $O_{AB}\subset O_{AD}$ for Riemann surface of genus $O$. Hence $|f(z)|<M$ and $R'\in O'_{AB}$ imply $D(f(z))<\infty$ and $R'\in O'_{AB}$. Thus we have Theorem 2.

6) $\overline{f(G)}$ means the closure of $f(G)$.