32. A Theorem for Metrizability of a Topological Space

By Jun-iti NAGATA
Department of Mathematics, Osaka City University
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Since Alexandroff and Urysohn's work various theorems concerning metrizability of a topological space were gotten by many mathematicians, but their methods of proofs are, in general, various and rather complicated. The purpose of this brief note is to prove a theorem for metrizability, which will contain a large number of metrizability theorems as direct consequences.[1]

We use the following theorem due to E. Michael[2] as well as the well-known theorem of P. Alexandroff and P. Urysohn.

Michael's theorem. A regular topological space R is paracompact if and only if every open covering of R has an open refinement $\mathfrak{V}=\bigcup_{n=1}^{\infty}\mathfrak{V}_n$, where each $\mathfrak{V}_n$ is a locally finite collection of open subsets of R.

Theorem 1. *In order that a T_1-topological space R is metrizable it is necessary and sufficient that one can assign a nbd (=neighborhood) basis $\{U_n(x)\mid n=1,2,\cdots\}$ for every point x of R such that for every n and each point x of R there exist nbds $S_n^1(x)$, $S_n^2(x)$ of x satisfying*

i) *$y\notin U_n(x)$ implies $S_n^2(y)\frown S_n^1(x)=\phi$,*

ii) *$y\in S_n^1(x)$ implies $S_n^2(y)\subseteq U_n(x)$.*

Proof. Since the necessity is clear, we prove only the sufficiency. To begin with, R is regular, since $\overline{S_n^1(x)}\subseteq U_n(x)$. Next, to show that R is paracompact, we take an arbitrary open covering $\mathfrak{V}=\{V_\alpha\mid \alpha<\tau\}$ of R.[3] If we let

$$V_{n\alpha}=\smile\{(S_n^1(x))^\circ\mid U_n(x)\subseteq V_\alpha\},^{4)}$$
$$V_{mn\alpha}=\smile\{U_m(x)\mid U_m(x)\subseteq V_{n\alpha}\},$$
$$V'_{mn\alpha}=\smile\{S_m^1(x)\mid U_m(x)\subseteq V_{n\alpha}\},$$
$$M_{mn\alpha}=(V'_{mn\alpha}-\smile_{\beta<\alpha}V_{n\beta})^\circ\quad(m,n=1,2,\cdots,\alpha<\tau),$$

then $\mathfrak{M}_{mn}=\{M_{mn\alpha}\mid\alpha<\tau\}$ is a locally finite open collection for each m,n. To show the local finiteness of $\mathfrak{M}_{mn}$ we choose, for an arbitrary point p of R, α $(\leqq\tau)$ such that $p\in V_{mn\alpha}$, $p\notin V_{mn\beta}$ $(\beta<\alpha)$. Then it follows from the condition i) of the proposition that $S_m^2(p)\frown V'_{mn\beta}=\phi$

1) The detail of the content of this note will be published in an another place.
2) See [3].
3) In this proof we denote by τ,α,β,γ ordinal numbers.
4) A° denotes the interior of A.

$(\beta<\alpha)$, which implies $S_m^2(p)\frown M_{mn\beta}=\phi$ $(\beta<\alpha)$. Since $p\in V_{mn\alpha}\subseteq V_{n\alpha}$ and $V_{n\alpha}$ is open, we obtain a nbd $V_{n\alpha}$ of p satisfying $V_{n\alpha}\frown M_{mn\gamma}=\phi$ $(\gamma>\alpha)$. Therefore the nbd $S_m^2(p)\frown V_{n\alpha}$ of p intersects at most one of elements of $\mathfrak{M}_{mn}$, proving the local finiteness of $\mathfrak{M}_{mn}$.

To assert that $\bigcup_{m,n=1}^{\infty}\mathfrak{M}_{mn}=\mathfrak{M}$ covers R, we consider an arbitrary point p of R. Let $p\in V_\alpha$, $p\notin V_\beta$ $(\beta<\alpha)$, $\alpha<\tau$, then we can choose n such that $U_n(p)\subseteq V_\alpha$. Since $p\in(S_n^1(p))^\circ\subseteq V_{n\alpha}$ for this n, we can choose m satisfying $U_m(p)\subseteq V_{n\alpha}$. In consequence we have $S_m^1(p)\subseteq V'_{mn\alpha}$. On the other hand, it follows from $p\notin V_\beta$ $(\beta<\alpha)$ and from i) that $S_n^2(p)\frown V_{n\beta}=\phi$ $(\beta<\alpha)$. This implies

$$S_m^1(p)\frown S_n^2(p)\subseteq V'_{mn\alpha}-\bigcup_{\beta<\alpha}V_{n\beta}$$

and consequently $p\in M_{mn\alpha}$, i.e. $\mathfrak{M}$ covers R. Since $\mathfrak{M}<\mathfrak{B}$ is obvious, we can conclude, from Michael's theorem, the paracompactness of R. Thus it follows from A. H. Stone's theorem[5] that R is fully normal.

To complete the proof, let us show that $\{S(p,\mathfrak{S}_m)\mid m=1,2,\cdots\}$[6] for $\mathfrak{S}_m=\{(S_m^2(y))^\circ\mid y\in R\}$ is a nbd basis of each point p of R. Let $U(p)$ be an arbitrary nbd of p, and choose n satisfying $U_n(x)\subseteq U(x)$, then we can find $m\geqq n$ with $U_m(x)\subseteq S_n^1(x)$. If $(S_m^2(y))^\circ\ni x$, then considering $S_m^2(y)\frown S_m^1(x)\neq\phi$, we have $y\in U_m(x)\subseteq S_n^1(x)$ from i). Hence it follows from ii) that $S_m^2(y)\subseteq S_n^2(y)\subseteq U_n(x)$ because we can assume, without loss of generality, that $m\geqq n$ implies $S_m^2(y)\subseteq S_n^2(y)$. Therefore we have $S(x,\mathfrak{S}_m)\subseteq U(x)$, i.e. $\{S(p,\mathfrak{S}_m)\mid m=1,2,\cdots\}$ is a nbd basis of p. Thus we conclude the metrizability of R by Alexandroff-Urysohn's theorem.

We enumerate some of direct consequences of this theorem without proofs.

Theorem 2 (Yu. Smirnov [8] and the author [6]). *A regular space R is metrizable if and only if there exists an open basis $\mathfrak{B}=\bigcup_{n=1}^{\infty}\mathfrak{B}_n$ of R, where each $\mathfrak{B}_n$ is a locally finite collection of open sets.*

Theorem 3 (K. Morita [5]). *A T_1-space R is metrizable if and only if there exists a countable collection $\{\mathfrak{F}_n\mid n=1,2,\cdots\}$ of locally finite closed coverings of R such that $S(x,\mathfrak{F}_n)\subseteq U(x)$ for any nbd $U(x)$ of any point x of R and for some n.*

Theorem 4 (R. H. Bing [1]). *A regular space R is metrizable if and only if there exists a countable collection $\{\mathfrak{U}_n\mid n=1,2,\cdots\}$ of open collections such that the sum of the closures of any subcollection of $\mathfrak{U}_n$ is closed and such that $\{S(p,\mathfrak{U}_n)\mid n=1,2,\cdots\}$ is a nbd basis of each point p of R.*

Theorem 5 (K. Morita [4]). *A generalization of Alexandroff-*

5) See [9].

6) $S(p,\mathfrak{S}_m)=\bigcup\{S\mid p\in S\in\mathfrak{S}_m\}$.

Urysohn's theorem. A T_1-space R is metrizable if and only if there exists a countable collection $\{\mathfrak{U}_n \mid n=1,2,\cdots\}$ of open coverings such that $\{S(S(p, \mathfrak{U}_n), \mathfrak{U}_m) \mid m, n=1,2,\cdots\}$ is a nbd basis of each point p of R.

Theorem 6 (A. H. Frink [2]). *A T_1-space R is metrizable if and only if one can assign a nbd basis $\{U_n(x) \mid n=1,2,\cdots\}$ for every point x of R such that for every n and $p \in R$ there exists $m=m(n,x)$ satisfying the condition: $U_m(x) \frown U_m(y) \neq \phi$ implies $U_m(y) \subseteq U_n(x)$.*

Theorem 7 (the author [7]). *A T_1-space R is metrizable if and only if there exists a family $\{f_\alpha \mid \alpha \in A\}$ of real valued continuous functions of R such that*

i) $\smile_{\beta \in B} f_\beta$ *and* $\frown_{\beta \in B} f_\beta$ *are continuous for every* $B \subseteq A$,

ii) *for any nbd $U(x)$ of any point x of R there exist $\alpha \in A$ and $\varepsilon > 0$ such that $f_\alpha(x) < \varepsilon$, $f_\alpha(y) \geqq \varepsilon$ $(y \notin U(x))$.*

Theorem 8. *A T_1-space R is metrizable if and only if there exists a non-negative function $\varphi(x,y)$ of $R \times R$ satisfying*

i) $\varphi(x,y)=\varphi(y,x)$

ii) $d(x,A)=inf\{\varphi(x,y) \mid y \in A\}$ *for every subset A of R is a continuous function of x,*

iii) $\{S_n(x) \mid n=1,2,\cdots\}$ *for $S_n(x)=\{y \mid \varphi(x,y)<1/n\}$ is a nbd basis of each point x of R.*[7]

References

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7) This theorem is an extension of Collorary 6 of [7]. An extension of Lemma 2 in that paper is, also, deduced from Theorem 1.