50. *Boundedness of Semicontinuous Finite Real Functions*

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A number of interesting characterizations of pseudo-compact spaces[1] has been given by various authors. In the present note, we concern with the spaces on which all semicontinuous finite valued real functions are bounded. All topological spaces to be considered in what follows will be assumed to satisfy the axiom T_1 of Fréchet.

THEOREM 1. *The following properties of a topological space E are equivalent:*

(1) *Every upper semicontinuous finite real function on E is bounded above.*[2]

(2) *Every lower semicontinuous finite real function on E is bounded below.*

(3) *The space E is countably compact.*[3]

Proof. It is clear that (1) and (2) are equivalent. To prove the implication (1)$\rightarrow$(3), suppose that E is not countably compact. Then there exists a sequence $\{x_n\}$ $(n=1, 2,\cdots)$ of points of E which has no cluster point. A set which consists of a single point being closed, the function f_n defined by

$$f_n(x)=\begin{cases} n & \text{for } x=x_n \\ 0 & \text{for } x\neq x_n \end{cases}$$

is upper semicontinuous for any n. As can readily be seen, the function $f(x)=\sum_{n=1}^{\infty} f_n(x)$ is of finite value. Since the subsequence $\{x_n\}_{n\geq m}$ is a closed set for any positive integer m, f is upper semicontinuous, but it is not bounded above. Conversely, let us suppose that the space E is countably compact, and let an upper semicontinuous finite real function f on E be given. Then, for any positive integer n, the set $O_n=\{x\in E; f(x)<n\}$ being open, the sets O_n form a countable open covering of E when n runs over the positive integers. Now, since E is countably compact, for a suitable positive integer m, we have $E=O_m$, that is to say $f(x)<m$ for all $x\in E$. This completes the proof of the theorem.

1) A completely regular space is said to be *pseudo-compact* if every continuous function on it is bounded.

2) We say that a real valued function f defined on a set E is *bounded above* (*bounded below*) if there is a constant k such that $f(x)\leq k$ $(f(x)\geq k)$ for all $x\in E$.

3) A topological space in which every countable open covering has a finite subcovering is called *countably compact*.

A family of subsets of a topological space E is said to be *point finite* if each point of E belongs at most to a finite number of the members of the family. A covering of E consisting of closed sets is called *closed covering* of E.

THEOREM 2. *The following properties of a topological space E are equivalent*:

(1) *Every upper semicontinuous finite real function on E is bounded below.*

(2) *Every lower semicontinuous finite real function on E is bounded above.*

(3) *Every point finite family of open sets of E consists of a finite number of the members.*

(4) *Every decreasing sequence of non-empty open sets of E has a non-empty intersection.*

(5) *Every sequence of open sets of E which possesses the finite intersection property has a non-empty intersection.*

(6) *Every countable closed covering of E has a finite subcovering.*

Proof. The proof of the implication (1)→(2) is straight-forward, and the equivalence of the properties (4), (5) and (6) may be shown easily. In order to prove that (2) implies (3), let a point finite family of open sets of E be given and suppose that it contains a sequence of open sets O_n $(n=1, 2,\cdots)$. Define

$$f_n(x)=\begin{cases} n & \text{for } x \in O_n \\ 0 & \text{for } x \bar{\in} O_n; \end{cases}$$

then each function f_n is lower semicontinuous. Since the family $\{O_n\}$ is point finite, the function $f(x)=\sup\limits_n f_n(x)$ is of finite value and lower semicontinuous; but, for any positive integer n, we can find a point $x \in E$ such that $f(x)>n$, contrary to (2). Therefore the family can not possess infinitely many members. The implication (3)→(4) is visible, since a decreasing sequence of open sets with empty intersection is point finite. Thus the proof of our theorem is completed if the implication (4)→(1) has been proved. Now, let f be an upper semicontinuous finite real function on E. For any positive integer n, the set $O_n=\{x \in E; f(x)<-n\}$ is open, so that O_n $(n=1, 2,\cdots)$ is a decreasing sequence of open sets. Therefore, if each open set O_n were not empty, so would be the intersection $\bigcap\limits_{n=1}^{\infty} O_n$; let $x \in \bigcap\limits_{n=1}^{\infty} O_n$, then for any positive integer n, we should have $f(x)<-n$, which is absurd. It follows that an open set O_n must be empty, and hence f is bounded below.

It is quite obvious that if a topological space has one of the properties mentioned in Theorem 2, then every open set of the space has also these properties. A topological space consisting of a finite number of points has of course these properties. On the other hand, we obtain the following

THEOREM 3. *A regular space E which has one of the properties mentioned in Theorem 2 consists of a finite number of points.*

Proof. Considering Theorem 2 of [3], we can immediately deduce from the preceding theorem that a regular space has one of the properties mentioned in Theorem 2 of the present note if and only if every open set of the space is countably compact. Hence, it suffices to prove the following proposition:

If E is a Hausdorff topological space, then every open set of E is countably compact if and only if E consists of finitely many points.

Proof. We have only to verify the only if part of the proposition. Suppose that the space E contains infinitely many points. Then we can extract a sequence $\{x_n\}$ from E. By the assumption, E being countably compact, the set A of cluster points of $\{x_n\}$ is not empty. Since A is closed, its complement A^c is open and consequently countably compact. Hence, the set A^c contains only a finite number of x_n's, since otherwise A^c would contain at least one cluster point of the sequence $\{x_n\}$. Thus all but a finite number of x_n's are cluster points of the sequence $\{x_n\}$. We may assume therefore, without loss of generality, that each member x_n is a cluster point of the sequence $\{x_n\}$. Let V_1 and U_2 be disjoint neighbourhoods of x_1 and x_2 respectively. Since x_2 is a cluster point of $\{x_n\}$, there exists in U_2 at least one of x_n's distinct from x_2; denote it by x_{n_3}. Then we can find in U_2 disjoint neighbourhoods V_2 and U_3 of x_2 and x_{n_3} respectively. Suppose now that we have constructed pairwise disjoint open sets $V_1, V_2, \cdots, V_m$ and U_{m+1} containing $x_1, x_2, x_{n_3}, \cdots, x_{n_{m+1}}$ respectively. We can find then in U_{m+1} at least one of x_n's distinct from $x_{n_{m+1}}$, say $x_{n_{m+2}}$, and we can assign to $x_{n_{m+1}}$ and $x_{n_{m+2}}$ disjoint neighbourhoods V_{m+1} and U_{m+2} respectively, where V_{m+1} and U_{m+2} are both contained in U_{m+1}. We obtain thus a subsequence $\alpha=\{x_1, x_2, x_{n_3}, \cdots, x_{n_m}, \cdots\}$ of the sequence $\{x_n\}$ and a sequence of pairwise disjoint open sets $\{V_m\}$ such that $x_1 \in V_1$, $x_2 \in V_2$ and $x_{n_m} \in V_m$ for $m=3, 4, \cdots$. Since the union $\bigcup_{m=1}^{\infty} V_m$ is open, it is countably compact by our assumption, but any cluster point of the sequence α can not belong to $\bigcup_{m=1}^{\infty} V_m$. We have thus reached a contradiction, and hence E must be a finite set.

A topological space is termed a *G_δ-space* provided each point of the space is a G_δ-set. As has been shown by F. W. Anderson [1], G_δ-spaces need not be regular.

THEOREM 4. *A G_δ-space E which has one of the properties mentioned in Theorem 2 is a finite set.*

Proof. Observe first that each point of the space E is an open set. In fact, each point x of E being a G_δ-set, there exists a sequence of open sets $\{O_n\}$ $(n=1, 2,\cdots)$ such that

$$\bigcap_{n=1}^{\infty} O_n=\{x\}.$$

Since the set consisting of a single point is closed, the intersection G_n of two sets, O_n and the complement of the point x, is open, and hence we have

$$\bigcap_{n=1}^{\infty} G_n=\phi \quad \text{(the empty set)}.$$

Therefore, by virtue of (5) of Theorem 2, there exists a finite number of G_n's, say $G_{n_1}, G_{n_2},\cdots, G_{n_m}$, such that

$$\bigcap_{i=1}^{m} G_{n_i}=\phi.$$

In other words, we have $\bigcap_{i=1}^{m} O_{n_i}=\{x\}$, and so the set $\{x\}$ is open. Now, if E were not a finite set, we could find a sequence $\{x_n\}$ of points of E with $x_n \neq x_m$ whenever $n \neq m$, $\{x_n\}$ being a point finite family of open sets; this contradicts (3) of Theorem 2. Thus E is a finite set.

Moreover, from the fact that a compact Hausdorff space is regular, it follows immediately that every metacompact[4] Hausdorff space having one of the properties mentioned in Theorem 2 is a finite set.

The author does not know however whether there exists a topological space which has the properties mentioned in Theorem 2 and has an infinite cardinal.

References

[1] F. W. Anderson: A lattice characterization of completely regular G_δ-spaces, Proc. Amer. Math. Soc., **6**, 757–765 (1955).

[2] R. Arens and J. Dugundji: Remark on the concept of compactness, Portugaliae Math., **9**, 141–143 (1950).

[3] K. Iséki and S. Kasahara: On pseudo-compact and countably compact spaces, Proc. Japan Acad., **33**, 100–102 (1957).

4) A topological space is said to be *metacompact* if every open covering of the space has a point finite refinement. Cf. R. Arens and J. Dugundji [2].