47. *On Open Mappings*

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1. Let X and Y be topological spaces and let f be a mapping of X onto Y. f is said to be open (closed) if the image of every open (closed) subset of X is open (closed) in Y. It is easy to see that if f is an open continuous mapping, then the local compactness is invariant under f. As a generalization of the notion of the local compactness, L. Zippin [4] has introduced the notion of the semicompactness.

In the present note, we shall consider conditions under which the semicompactness is invariant under an open continuous mapping. The conditions we obtain are sufficient; however, we show by an example that if we drop one of the conditions the semicompactness is not always invariant.

2. We begin with giving the definition of the semicompactness introduced by L. Zippin. A topological space X is called semicompact at a point x if every neighborhood U of x contains an open neighborhood V of x such that the boundary $\mathfrak{B}V$ is compact.[1] X is called semicompact if X has this property at every point.

Theorem 1. *Let f be an open continuous mapping of a topological space X onto a topological space Y. If f is closed, then the semicompactness is invariant under the mapping f.*

Proof. Let y be any point of Y and let $U(y)$ be any open neighborhood of y. Then $f^{-1}\{U(y)\}$ is an open set containing the inverse image $f^{-1}(y)$ since f is continuous. Let x be any point of $f^{-1}(y)$, then there exists an open neighborhood $O(x)$ such that $O(x)\subset f^{-1}\{U(y)\}$ and $\mathfrak{B}O(x)$ is compact since X is semicompact. Since f is an open and closed continuous mapping and $O(x)$ is an open set, $f\{O(x)\}$ is an open neighborhood of y such that $f\{O(x)\}\subset U(y)$ and $\mathfrak{B}f\{O(x)\}\subset f\{\mathfrak{B}O(x)\}$ (see G. T. Whyburn [2], p. 147). Since f is closed and continuous, $f\{\mathfrak{B}O(x)\}$ is a closed compact set in Y. Hence $\mathfrak{B}f\{O(x)\}$ is compact. Therefore Y is a semicompact space. Thus Theorem 1 is proved.

Theorem 2. *Let f be an open continuous mapping of a Hausdorff space X onto a weakly separable[2] Hausdorff space Y such that the inverse image $f^{-1}(y)$ is connected for every point y of Y. If the*

1) According to L. Zippin, we allow that $\mathfrak{B}V$ may be vacuous. A semicompact space is also said to be locally peripherally compact (cf. [1]).

2) A topological space satisfying the first axiom of countability will be called weakly separable.

boundary $\mathfrak{B}f^{-1}(y)$ *is compact for every point of* Y, *then the semicompactness is invariant under the mapping* f.[3)]

Proof. If we can prove that f is closed, Theorem 2 follows immediately from Theorem 1. So that we prove that f is closed in the following. Let y be any point of Y and let U be any open subset of X containing $f^{-1}(y)$ and let U_0 the union of all $f^{-1}(y')$ such that $f^{-1}(y')\subset U$. We shall prove that U_0 is an open set. Since X is semicompact, there exists an open neighborhood $V(x)$ of x for every point x of $\mathfrak{B}f^{-1}(y)$ such that $V(x)\subset U$ and $\mathfrak{B}V(x)$ is compact. By the assumption that $\mathfrak{B}f^{-1}(y)$ is compact, we can find a finite number of such $V(x)$ which covers $\mathfrak{B}f^{-1}(y)$, say $\{V(x_i)\}$ $(i=1,2,\cdots,m)$. Let $W=\bigcup_{i=1}^{m} V(x_i)\cup \operatorname{Int} f^{-1}(y)$, then W is an open neighborhood of $f^{-1}(y)$ and is contained in U. It is easily verified that $\mathfrak{B}W\subset \bigcup_{i=1}^{m}\mathfrak{B}V(x_i)\cup \mathfrak{B}f^{-1}(y)$. Hence $\mathfrak{B}W$ is compact. Suppose that $f^{-1}\{O(y)\}\not\subset W$ for every open neighborhood $O(y)$ of y. Since Y is a weakly separable space, there exists a basis $\{O_n(y)\}$ $(n=1,2,\cdots)$ for open neighborhoods of y such that $O_n(y)\supset O_{n+1}(y)$. Therefore there exists a positive integer n_0 such that $O_n(y)\subset f(W)$ for all $n\geqq n_0$. Hence $f^{-1}\{O_n(y)\}\cap W\neq\phi$ and $f^{-1}\{O_n(y)\}\cap CW\neq\phi$ where CW denotes the complement of W. Then we can find a sequence $\{y_n\}$ of points such that $y_n\in O_n(y)$ and $f^{-1}(y_n)\cap W\neq\phi$ and $f^{-1}(y_n)\cap CW\neq\phi$ $(n\geqq n_0)$. Then we have $f^{-1}(y_n)\cap\mathfrak{B}W\neq\phi$ for $n\geqq n_0$.

In fact, suppose on the contrary that $f^{-1}(y_n)\cap\mathfrak{B}W=\phi$, then we have $f^{-1}(y_n)=\{f^{-1}(y_n)\cap C\overline{W}\}\cup\{f^{-1}(y_n)\cap W\}$. Since $f^{-1}(y_n)\cap W\neq\phi$ and $f^{-1}(y_n)$ is connected by hypothesis, we get $f^{-1}(y_n)\cap C\overline{W}=\phi$. Hence $f^{-1}(y_n)\subset\overline{W}$. Therefore $f^{-1}(y_n)\cap\mathfrak{B}W\neq\phi$, but this contradicts the assumption that $f^{-1}(y_n)\cap\mathfrak{B}W=\phi$. Thus we get $f^{-1}(y_n)\cap\mathfrak{B}W\neq\phi$. Since $\mathfrak{B}W$ is compact, there exists a subsequence $\{y_{n_i}\}$ $(i=1,2,\cdots)$ of $\{y_n\}$ such that $x_{n_i}\in f^{-1}(y_{n_i})\cap\mathfrak{B}W$ and $\{x_{n_i}\}$ converges to a point x of $\mathfrak{B}W$. Then $\{y_{n_i}\}$ converges to the point $f(x)$. Since Y is a Hausdorff space, we get $y=f(x)$ and hence $x\in f^{-1}(y)$. Therefore x is an interior point of W. This contradicts that x belongs to $\mathfrak{B}W$. Therefore we can find an open neighborhood $O(y)$ such that $f^{-1}\{O(y)\}\subset W$. The above argument can be applied to any point y such that $f^{-1}(y)\subset U$. So that the set U_0 is an open set. Therefore, by G. T. Whyburn's theorem [3], f is a closed mapping. Thus the theorem is proved.

Remark. A. H. Stone [1] has proved that if a closed continuous mapping f of a topological space X onto a topological space Y satisfies the condition that $\mathfrak{B}f^{-1}(y)$ is compact and $f^{-1}(y)$ is connected for every

3) Cf. Theorem 2 in [1].

point y of Y, then the semicompactness is invariant under the mapping f. For the proof of Theorem 2, we can use Stone's result in stead of Theorem 1. From the proof of Theorem 2, we get also the following theorem.

Theorem 3. *Let f be an open continuous mapping of a semicompact Hausdorff space X onto a weakly separable Hausdorff space Y. If the inverse image $f^{-1}(y)$ is connected and the boundary $\mathfrak{B}f^{-1}(y)$ is compact for every point y of Y, then the mapping f is closed.*

3. In this section, we shall give two examples which show that we can not drop one of the restrictions imposed in Theorems 1 and 3.

Example 1. Let X be the set of points in the Euclidean plane such that $[(x_1, x_2) \mid 0 \leqq x_1, 0 \leqq x_2]$. We define the basis for open neighborhoods of each point (x_1, x_2) of X as follows. (i) The basis for open neighborhoods of the point (x_1', x_2') such that $x_1'^2 + x_2'^2 \neq 1$, $x_1'^2 + x_2'^2 \neq 0$, is the family of the sets of points (x_1'', x_2'') such that $(x_1' - x_1'')^2 + (x_2' - x_2'')^2 < \varepsilon$ and the circle $(x_1' - x_1)^2 + (x_2' - x_2)^2 = \varepsilon$ is disjoint with the arc $\widehat{AB}$ of the circle $x_1^2 + x_2^2 = 1$, where $A = (0, 1)$ and $B = (1, 0)$. (ii) Let $\alpha_1, \alpha_2, \cdots$ be the sequence of points on the arc $\widehat{AB}$ such that $\alpha_n \to B$ and $\widehat{\alpha_{n+1}B} \subsetneqq \widehat{\alpha_n B}$ for $n = 1, 2, \cdots$. Let $\beta_1, \beta_2, \cdots$ be the sequence of open arcs such that $\beta_1 = \widehat{AB} - \widehat{\alpha_1 B} - \{A\}$, $\beta_2 = \widehat{\alpha_1 B} - \widehat{\alpha_2 B} - \{\alpha_1\}$, $\beta_3 = \widehat{\alpha_2 B} - \widehat{\alpha_3 B} - \{\alpha_2\}, \cdots$. The basis for open neighborhoods of each point (x_1, x_2) on the arc $\widehat{AB}$ is the family of the open arcs contained in $\widehat{AB}$ to which (x_1, x_2) belongs. (iii) The basis for open neighborhoods of the point $(0, 0)$ is the family of the sets $\left[(x_1, x_2) \,\middle|\, x_1^2 + x_2^2 < \frac{1}{n}\right] \cup [(x_1, x_2) \mid n < x_1^2 + x_2^2] \cup [\beta_j \mid j \geqq n]$ where n is any positive integer greater than 1. Then it is easy to see that X is a semicompact topological space. Let Y be the set $[y \mid 0 \leqq y < 1] \cup [y \mid 1 < y < 2] \cup [\alpha_n^*, \beta_n^* \mid n = 1, 2, \cdots] \cup \{\alpha_0^*, \omega\}$. We define the basis for open neighborhoods of each point of Y as follows. (1) For the point y such that $0 < y < 1$ or $1 < y < 2$, all open intervals with center y contained in $0 < y < 1$ or $1 < y < 2$. (2) For the point $y = 0$, all sets $\left[y \,\middle|\, 0 \leqq y < \frac{1}{n}\right] \cup \left[y \,\middle|\, 2 - \frac{1}{n} < y < 2\right] \cup [\beta_j^* \mid j \geqq n]$, where n is any positive integer greater than 1. (3) For each point β_n^*, the set $\{\beta_n^*\}$. (4) For each point α_n^* $(n = 1, 2, \cdots)$, the set $\{\alpha_n^*, \beta_n^*, \beta_{n+1}^*\}$. (5) For the point ω, all sets $[\alpha_j^*, \beta_j^* \mid j \geqq n]$ where $n = 1, 2, \cdots$. (6) For the point α_0^*, the set $\{\alpha_0^*, \beta_1^*\}$. Then we can easily verify that Y is a topological space and is not semicompact.

Now we shall define a mapping f of X onto Y. Let p be the point (x_1, x_2). If $x_1^2 + x_2^2 = y < 1$ or $x_1^2 + x_2^2 = y > 1$, then $f(p) = y$ or $f(p) = 2 - \frac{1}{y}$. If $p \in \beta_n$ or $p = \alpha_n$, then $f(p) = \beta_n^*$ or $f(p) = \alpha_n^*$. And let

$f(A)=\alpha_0^*$ and let $f(B)=\omega$. Then it is easy to see that f is an open continuous mapping, but is not closed. Therefore by this example, we can see that in Theorem 1, if we drop the condition that f is closed, the semicompactness is not always invariant under f.

Example 2. Let X be the set of points (x_1, x_2) in the Euclidean plane such that $0 \leqq x_1 \leqq 1$, $0 < x_2 < 1$. Then the subspace X of the Euclidean plane is a semicompact (locally compact) Hausdorff space. Let Y be the closed interval $[0, 1]$ in the real number space. Then Y is a weakly separable Hausdorff space. Let f be a mapping of X onto Y such that $f(p)=x_1$ for every point $p=(x_1, x_2)$ of X. Then it is obvious that f is an open continuous mapping such that $f^{-1}(y)$ is connected but $\mathfrak{B}f^{-1}(y)$ is not compact. Furthermore it is easy to verify that f is not closed. Therefore we can see that in Theorem 3, if we drop the condition that $\mathfrak{B}f^{-1}(y)$ is compact for every point y of Y, the mapping f is not always closed.

References

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