83. *On Kählerian Manifolds with Positive Holomorphic Sectional Curvature*

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S. B. Myers [2] proved that any complete Riemannian manifold with positive sectional curvature is compact. And J. L. Synge [3] proved that if an orientable even dimensional Riemannian manifold is complete and has positive sectional curvature, then it is simply connected. The purpose of this note is to generalize these results to Kählerian manifolds*) with positive holomorphic sectional curvature.

A Kählerian manifold M is said to have positive holomorphic sectional curvature greater than or equal to a positive number e if the holomorphic sectional curvature of M is greater than or equal to e at all points and for all directions. In this note we consider a real representation of Kählerian manifold, that is, the Riemannian manifold with the parallel complex structure.

Theorem 1. A complete n-dimensional Kählerian manifold M whose holomorphic sectional curvature is greater than or equal to $e>0$ is compact and has the diameter less than or equal to $\pi/\sqrt{e}$.

Proof. Let I be the tensor of type (1, 1) which defines the complex structure of M. Let A and B be any two points of M and g be a minimizing geodesic arc which binds A and B. Let X_A be the unit tangent vector to g at A. We can obtain the parallel vector field X on g by displacing X_A parallely along g.

Since I is a parallel tensor field on M, IX is parallel and moreover it is perpendicular to X on g. Let p be an arbitrary point on g and π_p be the tangent subspace of M at p which is spanned by X_p and $(IX)_p$.

Now we can construct a 2-dimensional surface S in a neighborhood of g which has π_p as tangent space at p, $p\in g$.

We induce Riemann metric of M to S. Then, according to Synge's theorem (cf. Chern [1, p. 137]), the Gaussian curvatures of S are equal at all points of g to the sectional curvatures in the corresponding tangent plane π_p of S. So these Gaussian curvatures are greater than or equal to e.

By Sturm's comparison theorem (cf. Myers [2]), the length of g can not be greater than $\pi/\sqrt{e}$. So the diameter of M is less than or equal to e and hence M is compact.

Q.E.D.

*) They hold good for pseudo-Kählerian manifolds, too.

Theorem 2. If a Kählerian manifold M with positive holomorphic sectional curvature is complete, there does not exist a closed geodesic of minimum type.

Proof. Suppose that there exists such a closed geodesic g. In a neighborhood of g, we introduce a system of Fermi's coordinates (x^i) and construct 2-dimensional surface S passing through g in the same way as in the discussion of Theorem 1.

Now we introduce on S a system of Fermi's coordinates (u, v) around g, i.e.

v: the length of geodesic on S perpendicular to g;

u: the length of arc of g from a fixed origin.

Then g is given by the equation $v=0$.

Now we consider as variational curves closed curves $v=$constant.

The length of these closed curves depends on v. So we denote it by $L(v)$. If we denote Riemann metric of M by $g_{ij}dx^i dx^j$ and the length of g by s, then we have

$$L(v)=\int_0^s \sqrt{g_{ij}\frac{\partial x^i}{\partial u}\frac{\partial x^j}{\partial u}}\,du.$$

By a straight forward calculation (cf. Preismann [4]) we get

$$L'(0)=0, \qquad L''(0)=\int_0^s -K\,du, \tag{1}$$

where K is the Gaussian curvature of S on g and the dash denotes the differentiation with respect to v.

Since $K>0$ on g, we have

$$L''(0) < 0. \tag{2}$$

By (1) and (2), g can not be a closed geodesic of minimum type. This contradicts the assumption.

Q.E.D.

Theorem 3. A complete n-dimensional Kählerian manifold M with positive holomorphic sectional curvature is simply connected.

Proof. Assume the theorem is false.

By Theorem 1, M is compact. Then it is well known (cf. Chern [1, p. 156] or Preismann [4]) that in every free homotopy class $\neq 0$ of closed curves of M there is a closed geodesic which is the shortest closed curve in the class. Since M is not simply connected, there exists a non-trivial free homotopy class and hence a non-trivial shortest closed geodesic. This is a closed geodesic of minimum type.

This contradicts the assumption.

Q.E.D.

The following fact is well known.

Corollary. A complex projective space is compact and simply connected.

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References

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