93. Pseudo-compactness and μ-convergence

By Kiyoshi ISÉKI

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Following E. Hewitt, a completely regular space is said to be *pseudo-compact*, if every real continuous function on it is bounded. In a paper [2] of K. Iséki a characterisation of pseudo-compact spaces was given in terms of uniformly convergences and its related concepts. In this Note, we shall give a further characterisation by the μ-convergence of G. Sirvint [6].

A sequence $\{f_n(x)\}$ of real functions defined in an abstract set S is said to be *quasi-uniformly convergent* to $f(x)$ on S if it converges to $f(x)$ and if, for given positive ε and integer N, there is a finite number of indices $n_1, n_2, \cdots, n_k \geq N$ such that for each x of S, at least one of the following relations holds:

$$|f_{n_i}(x)-f(x)|<\varepsilon, \quad i=1, 2, \cdots, k.$$

Then we have the following

Lemma 1. If a sequence $\{f_n(x)\}$ of real continuous function of a pseudo-compact space S is convergent to 0, *then it converges quasi-uniformly to* 0 *on S.*

Proof. For a given positive ε and a given integer N, we shall define the sets O_n as

$$O_n=\{x\,|f_n(x)|<\varepsilon\}, \quad n=N, N+1, \cdots .$$

Since $f_n(x)$ is continuous, each $\{O_n\}$ is open, and from $f_n(x)\to 0$ on S, $\{O_n\}$ is a countable open covering of S. Therefore, by a theorem of S. Mardešić and P. Papić [5], there is a finite set of indices $n_1, \cdots, n_k$ such that $\bigcup_{i=1}^{k} \overline{O}_{n_i} \supset S$. Therefore $f_n(x)$ converges quasi-uniformly to 0.

Conversely, we shall show the following

Lemma 2. Let S be a completely regular space. If a sequence $\{f_n(x)\}$ of continuous functions on S which converges to 0 *converges quasi-uniformly to* 0, *then S is pseudo-compact.*

Proof. Suppose that there is an unbounded continuous function $f(x)$ on S. Then we can find a sequence $\{x_n\}$ such that $x_n \in S$ and $|f(x_n)|\to\infty$ $(n\to\infty)$. Let $f_n(x)=\dfrac{f(x)}{f(x_n)}$, then we have $f_n(x)\to 0$ $(n\to\infty)$ on S (pointwise convergence!). For a given N and $N<m$, we have

$$|f_{n_1}(x_{n_2})|=\left|\frac{f(x_{n_2})}{f(x_{n_1})}\right|>1$$

for $N\leq n_1 \leq m < n_2$. Since m may be taken arbitrary, this implies that the sequence $\{f_n(x)\}$ is not quasi-uniformly convergent. Q.E.D.

K. Iséki [2] obtained a characteristic property of a pseudo-compact space as follows: *A completely regular space S is pseudo-compact, if and only if a monotone sequence of continuous functions which converges to a continuous function is convergent uniformly on S.* Therefore, in a pseudo-compact space, if a decreasing sequence of continuous function converges to 0, then it converges uniformly to 0.

Following G. Sirvint [6], we shall say that a sequence $\{f_n(x)\}$ of real functions on an abstract space S *μ-converges* to 0 on S, if, for a given positive ε, we can find non-negative numbers $\lambda_1,\cdots,\lambda_n$ such that

$$\sum_{i=1}^{n}\lambda_i=1, \quad \left|\sum_{i=1}^{n}\lambda_i f_i(x)\right|<\varepsilon \quad \text{on } S.$$

Hence, *if a decreasing sequence $f_n(x)$ of continuous functions on a pseudo-compact space S converges to 0, then it is μ-convergent to 0 on S.*

Conversely, we shall show that this property characterizes the notion of pseudo-compactness.

To do so, suppose that there is an unbounded continuous function $f(x)$ on a completely regular space S. Then we can find a sequence $\{x_n\}$ of S such that $|f(x_n)|<|f(x_{n+1})|$ $(n=1,2,\cdots)$ and $|f(x_n)|\to\infty$ $(n\to\infty)$.

Let $f_n(x)=\dfrac{f(x)}{|f(x_n)|}$, then we have $f_n(x)\to 0$ $(n\to\infty)$ on S and $\{f_n(x)\}$ is a decreasing sequence. For any $\lambda_i\geq 0$ such that $\sum_{i=1}^{n}\lambda_i=1$, we have

$$\sum_{i=1}^{n}\lambda_i f_i(x_n)=\sum_{i=1}^{n}\lambda_i\frac{f(x_n)}{f(x_i)}\geq\sum_{i=1}^{n}\lambda_i=1.$$

Therefore $\{f_n(x)\}$ is not μ-convergent to 0 on S. Hence we have the following

Theorem. For a completely regular space S, the following conditions are equivalent:

(1) *S is pseudo-compact.*

(2) *If a sequence of continuous functions converges to 0, then it converges quasi-uniformly to 0 on S.*

(3) *If a decreasing sequence of continuous functions converges to 0, then it is μ-convergent to 0 on S.*

(4) *Any sequence of continuous functions which converges to a continuous function is convergent to the function quasi-uniformly on S.*

(5) *Any sequence of continuous functions which converges simply-uniformly to a function at every point of S is quasi-uniformly convergent to the function at each point of S.*

As to the propositions (4) and (5), by the proofs of Lemmata 1, and 2, (4) is equivalent to (1). Such a characterisation for countably compact normal space was obtained by T. Isiwata [3]. Therefore the proposition (4) is a generalisation of his Theorem 2. The proposition (5) is obtained by K. Iséki [2, Theorem 6] and (4).

In my short Note, *A note on compact space*, Proc. Japan Acad., 33, 271 (1957), the present writer gave a remark. *Any pseudo-compact complete uniform space is compact.* Therefore *if the space S of Theorem is complete uniform space, each proposition of Theorem gives a characterisation of compact space.*

In their paper [4], G. Fichtenholz and L. Kantorovitch introduced a concept of convergence, *almost uniformly convergence.* A sequence $\{f_n(x)\}$ of functions defined on a set S is said to be *almost uniformly convergent* to $f(x)$ on S, if it converges quasi-uniformly to $f(x)$ on S, together with any partial sequence.

From Theorem, we have the following

Corollary 1. Let S be a pseudo-compact space, and suppose that a sequence of continuous functions converges to a continuous function on S. Then the convergence is almost uniformly.

The following proposition is due to essentially R. G. Bartle [1, Theorem 7.1].

Proposition. Let A be a dense subset of a pseudo-compact space S and suppose that a sequence of continuous functions converges to a continuous function at every point of A. Then the sequence converges to the continuous function at every point of S, if and only if the convergence is almost uniformly on A.

The necessity of proposition follows from Theorem. The proof of the sufficiency is the same with R. G. Bartle [1, Theorem 7.1]. Therefore, we shall omit the proof.

Let $\beta(S)$ be the Čech compactification of a completely regular space S, and suppose that a sequence $\{f_n(x)\}$ of bounded continuous functions which converges to a bounded continuous function $f(x)$ on S, and let f_n^*, f^* be the extensions of f_n, f on $\beta(s)$ respectively. If S is pseudo-compact, the convergence $f_n(x)\rightarrow f(x)$ is almost uniformly, and by Proposition or Theorem 7.1 of R. G. Bartle [1], we have $f_n^*(x)$ converges to $f^*(x)$ on $\beta(S)$. Therefore we have

Corollary 2. Let S be a pseudo-compact, and let $f_n(x)$ and $f(x)$ be continuous functions such that $f(x)$ converges to $f(x)$ on S. If $f_n^*(x)$ and $f^*(x)$ are the extensions of $f_n(x)$ and $f(x)$ on the Čech compactification $\beta(S)$ of S respectively, then $f_n^*(x)$ converges to $f^*(x)$ on $\beta(S)$.

For the related theorem for a normal space, see T. Isiwata [3, p. 187].

References

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*) Theorem 7 in this paper should be read as follows: *A normal space S is countably compact, if and only if every sequence of continuous functions which is simply-uniformly convergent converges quasi-uniformly.*