92. AU-covering Theorem and Compactness

By Kiyoshi ISÉKI
Kobe University

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To characterize a countably compact normal space, the present author introduced the concept of *AU*-property of covering [2-7]. In this paper, we shall discuss the *AU*-covering theorem of a topological space.

Let S be a topological space, and let Φ be an open covering of S. By $\overline{\overline{\Phi}}$, we denote the power of the set of elements of Φ. A covering Φ is said to be *AU-reducible*, if it contains a subfamily Ψ of lower power than $\overline{\overline{\Phi}}$ such that the union of the closures of elements of Ψ is S, otherwise it is *AU-irreducible*.[1]

For our discussion, we shall use some classical concepts which were introduced by E. W. Chittenden [1].

A covering Φ with power $\aleph_\mu$ is normal, if it can be well-ordered in the form

$$O_1, O_2, \cdots, O_\alpha, \cdots \quad (\alpha < \omega_\mu)$$

where ω_μ is the initial ordinal of $\aleph_\mu$ and, for each α, there is an element p_α such that $p_\alpha \in \overline{O}_\alpha$ and $p_\alpha \bar{\in} \overline{O}_\beta$ $(\beta<\alpha)$. The set $\{p_\alpha \mid \alpha<\omega_\mu\}$ is said to be an *associate set* of the normal covering Φ. It is clear that *every normal covering is AU-irreducible*. Now we shall prove the following fundamental

Theorem 1. Every open covering Φ contains a normal covering.

Proof. Let Φ_1 be an *AU*-irreducible subcovering of Φ, and let

$$(1) \qquad O_1, O_2, \cdots, O_\mu, \cdots \quad (\alpha<\omega_\mu)$$

be a well-ordered set of the type ω_μ of Φ_1. To construct a normal subfamily Ψ of Φ_1, let $O_{\alpha_1}=O_1$, and take $p_2 \in S-\overline{O}_{\alpha_1}$, let O_{α_2} be the first term of the transfinite sequence (1) such that $O_\alpha \ni p_2$. Suppose that O_{α_ν} $(\nu<\beta)$ is defined, then we shall define O_{α_β} as follows: take $p_\beta \in S-\bigcup_{\nu<\beta} \overline{O}_{\alpha_\nu}$, let O_{α_β} be the first term of (1) such that $\overline{O}_\alpha \ni p_\beta$. Since Φ_1 is irreducible, α_β $(\beta<\omega_\mu)$ is defined and $\{\alpha_\beta\}$ is cofinal with $\{\alpha \mid \alpha<\omega_\mu\}$. Therefore $\Psi=\{O_{\alpha_\beta} \mid \beta<\omega_\mu\}$ is a normal covering of power $\aleph_\mu$. Q.E.D.

If an open covering Φ is locally finite (or star finite), then $\overline{\Phi}$ consisting of the closures of all elements of Φ is locally finite (or star finite) and the union of the closures of some elements of Φ is closed.

1) For the usual concept of reducibility, see E. W. Chittenden [1].

Therefore as each element p_α of an associated set of a normal locally finite covering, we can choice a family of disjoint open sets G_{α_β} $(\beta<\omega_\mu)$. If S is regular, we can take the family G_{α_β} $(\beta<\omega_\mu)$ as closure disjoint. Hence we have the following

Lemma 1. There is a closure pairwise disjoint associated open set family for any normal locally finite (or star finite) open covering in a regular space.

Therefore, if S is regular, we shall use such an associated open set family stated in Lemma 1.

Let $\{O_n \mid n=1,2,\cdots\}$ be any normal open covering consisting of countable members in a regular space, then we can take a closure disjoint associated open set family. Therefore we have the following

Lemma 2. There is a closure disjoint associated open set family for any countable normal open covering in a regular space.

We shall give some propositions on a weakly compact regular space by using Lemmas 1 and 2. Such a concept was introduced by S. Mardešić and P. Papić [11]. Any topological space is said to be *weakly compact*, if for every family of disjoint open sets $\{O_\alpha\}$, $\underline{\lim}\, O_\alpha \neq \phi$ in a topological sense. S. Mardešić and P. Papić [11], the present writer [3-7] and S. Kasahara [8] studied such a weakly compact regular space.[2)]

For any weakly compact regular space S, let $\{O_\alpha\}$ be a locally finite open covering of S, then, by Theorem 1, we can find a normal subfamily of it. By Lemma 1, we can take an associated open sets family Φ. If Φ has infinitely many members, by the weakly compactness, some member of Φ must intersect the closure of some other one of it. Therefore there is a normal subfamily having finite members. By a similar method and Lemma 2, we have the following

Theorem 2. For any regular space S the following conditions are equivalent:

1) *S is weakly compact.*

2) *Any locally finite open covering contains an AU-covering.*

3) *Any countable infinite open covering contains an AU-covering, i.e. AU-reducible.*

In a regular space, it is clear that the condition 1) or the condition 2) implies the weakly compactness (see K. Iséki [5] or S. Mardešić and P. Papić [11]).

Next, we shall consider the *classical normal family* in the sense of E. W. Chittenden [1, p. 299]. We shall take a special type in

2) A few weeks ago, we received a recent volume of Bull. Amer. Math. Soc., 63/1 (1957). We found abstracts by C. Wenjen, J. D. McKnight, R. W. Bagley and E. H. Connell. They has also studied a weakly compact space. The abstracts contain some of the results which we have published in the recent volume of Proc. Japan Acad. (1957).

which the *relation T* means "*interior to*". Every locally finite open covering Φ of a space S contains a classical normal covering ψ and ψ is a locally finite covering of S. Let $\bar{\bar{\psi}}=\aleph_\mu$ and let $\{p_\alpha\mid\alpha<\omega_\mu\}$ be an associated set of ψ. Suppose that

(2) $$O_1, O_2,\cdots, O_\alpha,\cdots \quad (\alpha<\omega_\mu)$$

is a well-ordered series of ψ with the associated set $\{p_\alpha\mid\alpha<\omega_\mu\}$. Since ψ is locally finite, there is a neighbourhood of p_1 meeting an only finite members of (2). Therefore, there is a neighbourhood U_1 of p_1 not containing p_α $(\alpha\neq 1)$. If S is regular, we can find an open set V_1 such that $p_1\in V_1\Subset U_1$ and the open set V_1 does not contain p_α $(\alpha\neq 1)$. For the point p_2, we take a neighbourhood U_2 of p_2 not meeting $\bar{V}_1$ and not containing p_α $(\alpha\neq 2)$ and $U_2\subset Q_2$. By the regularity, we can find an open set V_2 such that $p_2\in V_2\Subset U_2$. In general, suppose that $\{\nu_\beta\mid\beta<\alpha\}$ is defined, we shall define V_α as follows: Since the family $\{V_\beta\mid\beta<\alpha\}$ is locally finite, $\overline{\bigcup_{\beta<\alpha} V_\beta}=\bigcup_{\beta<\alpha}\bar{V}_\beta$ and $p_\alpha\in \bar{V}_\beta$ $(\beta<\alpha)$. Hence we take a neighbourhood U_α of p_α such that it does not meet $\bigcup_{\beta<\alpha}\bar{V}_\beta$ and is contained in O_α and does not contain p_γ $(\gamma\neq\alpha)$. Take an open set V_α such that $p_\alpha\in V_\alpha\Subset U_\alpha$. V_α is a required open set. Therefore, we have a well-ordered open sets V_α of type ω_μ and this is a closure disjoint set family. Therefore, with Theorem 2, we have the following

Theorem 3. A regular space is weakly compact if and only if every locally finite open covering contains a finite subcovering.

Theorem 3 is also found in S. Kasahara [8].

Let $\{F_\alpha\}$ be a given family of closed sets with finite intersection property in a compact space S. If, for some open set U, $\bigcap_\alpha F_\alpha\subset U$, then we have

$$\bigcup_\alpha(S-F_\alpha)\supset S-U$$

and since $S-U$ is closed and S is compact, there are finite set of indices $\alpha_1,\cdots,\alpha_n$ such that

$$\bigcup_{i=1}^{n}(S-F_{\alpha_i})\supset S-U.$$

Therefore we have $\bigcap_{i=1}^{n}F_{\alpha_i}\subset U$. Conversely, let $\{F_\alpha\}$ be a family of closed sets with finite intersection property. If $\bigcap_\alpha F_\alpha=\phi$, then since empty set is open, there are finite set of indices $\alpha_1,\cdots,\alpha_n$ such that

$$\bigcap_{i=1}^{n}F_{\alpha_i}=\phi.$$

Therefore this leads to a contradiction. Hence we have the following

Proposition 1. A topological space is compact, if and only if, for every family of closed sets F_α *with finite intersection property such that* $\bigcap_\alpha F_\alpha \subset U$, *there is a finite set of indices* $\alpha_1, \cdots, \alpha_n$ *such that* $\bigcap_{i=1}^{n} F_{\alpha_i} \subset U$, *where* U *is open set.*

By a similar method, we can prove the following

Proposition 2. A topological space is countably compact, if and only if, for every decreasing sequence of closed sets F_n, $\bigcap F_n \subset U$ *implies* $F_{n_0} \subset U$ *for some* n_0, *where* U *is open.*

To formulate such propositions to a weakly compact regular space, we shall use regularly open sets (for the definition and its related theorems, see C. Kuratowski [9]). Then we have the following

Theorem 4. A necessary and sufficient condition that a regular space S *be weakly compact is that for any regularly open set* U *containing the intersection of a decreasing sequence of closed sets* F_n, *there is a closed set* F_{n_0} *such that the interior of* F_{n_0} *is contained in* U.

Proof. Let S be a weakly compact regular space. Then, $\bigcap_{n=1}^{\infty} F_n \subset U$ implies $\bigcup_{n=1}^{\infty}(S-F_n) \supset S-U$ and since U is regularly open, $S-U$ is regularly closed. Hence $S-U$ is the closure of some open set and $\{S-F_n\}$ $(n=1, 2, \cdots)$ is an open covering of $S-U$. By Theorem 4 of my paper [6], $S-U$ is weakly compact. Hence there is a closed F_{n_0} such that $\overline{S-F_{n_0}} \supset S-U$. Hence $\mathrm{Int}\, F_{n_0} = S - \overline{S-F_{n_0}} \subset U$, which completes the necessity. To prove the converse, let $\{F_n\}$ be a decreasing sequence of closed sets with non-empty interiors, suppose that $\bigcap_{n=1}^{\infty} F_n = \phi$. The empty set is regular open, so we have $\mathrm{Int}\, F_{n_0} = \phi$ for some n_0 by the hypothesis. Hence $\bigcap_{n=1}^{\infty} F_n$ is non-empty set. Therefore, by Theorem 2 of my paper [6], S is weakly compact.

The closure of an open set in an absolute closed Hausdorff space is absolute closed (see M. Katětov [10]). Therefore, for an absolute closed Hausdorff space, we have the following

Theorem 5. A Hausdorff space is absolute closed if and only if, for every transfinite decreasing sequence of closed sets F_α *with non-empty interiors and a regularly open set* U, $\bigcap_\alpha F_\alpha \subset U$ *implies* $\mathrm{Int}\, F_{\alpha_0} \subset U$ *for some* α_0.

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