151. Note on Free Products

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(Comm. by K. SHODA, M.J.A., Dec. 12, 1957)

In the notes [1] and [2], we have studied the necessary and sufficient condition for the existence of the free algebraic systems, and the other results. In this note, a free P-product which contains, as the special case, the free A-product of A-algebraic systems will be defined in the similar way as the free A-product has been defined by K. Shoda [3]. And we shall show a necessary and sufficient condition that a free P-product of P-algebraic systems $\mathfrak{A}_1,\cdots,\mathfrak{A}_n$ is an extension of the P-algebraic systems $\mathfrak{A}_1,\cdots,\mathfrak{A}_n$.

Let V be a system of single-valued compositions—hereafter every algebraic system and every composition-identity will be considered with respect to V. Let $A(x_1,\cdots,x_r)$ and $B(x_1,\cdots,x_r)$ be two sets of composition-identities of variables $x_1,\cdots,x_r$, and let $\mathfrak{A}$ be an algebraic system. If the elements $a_1,\cdots,a_r$ in $\mathfrak{A}$ satisfy all the composition-identities of $A(x_1,\cdots,x_r)$, we say that the elements $a_1,\cdots,a_r$ satisfy $A(x_1,\cdots,x_r)$, and denote it by $A[a_1,\cdots,a_r]$. An algebraic system $\mathfrak{A}$ is said to satisfy an implication $A(x_1,\cdots,x_r)\Rightarrow B(x_1,\cdots,x_r)$, when any elements $a_1,\cdots,a_r$ in $\mathfrak{A}$ satisfy the following condition: If $A[a_1,\cdots,a_r]$, then $B[a_1,\cdots,a_r]$. Now let P be a family of implications $A_\kappa(x_1,\cdots,x_{r_\kappa})\Rightarrow B_\kappa(x_1,\cdots,x_{r_\kappa})$, and let $\{a_\lambda\mid\lambda\in L\}$ be a system of generators. Then we can define P-algebraic systems generated by the system $\{a_\lambda\mid\lambda\in L\}$ of generators. Moreover, by Theorem 3 in [1], there exists a free P-algebraic system $F(\{a_\lambda\mid\lambda\in L\},P,R)$ with any set R of relations, since the implication $A_\kappa(x_1,\cdots,x_{r_\kappa})\Rightarrow B_\kappa(x_1,\cdots,x_{r_\kappa})$ can be considered as a set of implications in the sense of the note [1].

Let $\mathfrak{A}$ and $\mathfrak{B}$ be any two P-algebraic systems. Then it is clear from Theorem 1 in [1] that $\mathfrak{A}$ and $\mathfrak{B}$ can be denoted by $F(\{a_\lambda\mid\lambda\in L\},P,R)$ and $F(\{b_\mu\mid\mu\in M\},P,S)$ respectively. The P-algebraic system $F(\{a_\lambda\mid\lambda\in L\}\smile\{b_\mu\mid\mu\in M\},P,R\smile S)$ is called a free P-product of $\mathfrak{A}$ and $\mathfrak{B}$, and is denoted by $\mathfrak{A}*\mathfrak{B}$. Then there always exists a free P-product of any two P-algebraic systems $\mathfrak{A}$ and $\mathfrak{B}$ by Theorem 3 in [1], and it is easy to see that the free P-product $\mathfrak{A}*\mathfrak{B}$ is uniquely determined, i.e. $\mathfrak{A}*\mathfrak{B}$ does not depend on the choice of the generator systems $\{a_\lambda\mid\lambda\in L\}$ and $\{b_\mu\mid\mu\in M\}$. A free P-product of any number of P-algebraic systems $\mathfrak{A}_1,\cdots,\mathfrak{A}_n$ can be similarly defined. A P-extension of a P-algebraic system $\mathfrak{A}$ will always mean a P-algebraic system which contains $\mathfrak{A}$ as a subsystem.

Theorem 1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two P-algebraic systems. Then, in order that $\mathfrak{A}$ is contained in the free P-product of $\mathfrak{A}$ and $\mathfrak{B}$, it is necessary and sufficient that there exists a P-extension $\mathfrak{A}^*$ of $\mathfrak{A}$ such that a homomorphism of $\mathfrak{B}$ into $\mathfrak{A}^*$ exists.*

Proof. By Theorem 1 in [1], $\mathfrak{A}$ and $\mathfrak{B}$ can be denoted by $F(\{a_\lambda \mid \lambda \in L\}, P, R)$ and $F(\{b_\mu \mid \mu \in M\}, P, S)$ respectively. Now suppose that $\mathfrak{A}$ is contained in the free P-product $\mathfrak{A}*\mathfrak{B}=F(\{a_\lambda \mid \lambda \in L\} \smile \{b_\mu \mid \mu \in M\}, P, R \smile S)$. Then, a subsystem $\mathfrak{B}'$ of $\mathfrak{A}*\mathfrak{B}$ which is generated by the set $\{b_\mu \mid \mu \in M\}$ can be denoted by $F(\{b_\mu \mid \mu \in M_J, P, S')$ such that S' contains S. Hence $\mathfrak{B}'=F(\{b_\mu \mid \mu \in M\}, P, S')$ is homomorphic to $\mathfrak{B}=F(\{b_\mu \mid \mu \in M\}, P, S)$ by Theorem 2 in [1]. Therefore, if we put $\mathfrak{A}^*=\mathfrak{A}*\mathfrak{B}$, then there exists a homomorphism of $\mathfrak{B}$ into $\mathfrak{A}^*$. This completes the proof of the necessity. In the following, we shall prove the sufficiency. Suppose that there exists a P-extension $\mathfrak{A}^*=F(\{a_\nu^* \mid \nu \in N\}, P, R^*)$ of $\mathfrak{A}$ such that a homomorphism φ of $\mathfrak{B}$ into $\mathfrak{A}^*$ exists. Then we have

$$
\begin{aligned}
\mathfrak{A}^* * \mathfrak{B} &= F(\{a_\nu^* \mid \nu \in N\}, P, R^*) * F(\{b_\mu \mid \mu \in M\}, P, S) \\
&= F(\{a_\nu^* \mid \nu \in N\} \smile \{b_\mu \mid \mu \in M\}, P, R^* \smile S) \\
&\gtrsim F(\{a_\nu^* \mid \nu \in N\} \smile \{b_\mu \mid \mu \in M\}, P, R^* \smile S \smile \{b_\mu = \varphi(b_\mu) \mid \mu \in M\}) \\
&= F(\{a_\nu^* \mid \nu \in N\}, P, R^*) = \mathfrak{A}^*.
\end{aligned}
$$

Hence $\mathfrak{A}^* * \mathfrak{B}$ contains $\mathfrak{A}^*$, and hence $\mathfrak{A}^* * \mathfrak{B}$ contains $\mathfrak{A}$. Now let $\mathfrak{C}$ be the subsystem of $\mathfrak{A}^* * \mathfrak{B}$ which is generated by the set $\{a_\lambda \mid \lambda \in L\} \smile \{b_\mu \mid \mu \in M\}$. Then $\mathfrak{A}$ is contained in $\mathfrak{C}$, and the subsystem $\mathfrak{C}$ can be denoted by $F(\{a_\lambda \mid \lambda \in L\} \smile \{b_\mu \mid \mu \in M\}, P, T)$ such that T contains R and S. Therefore we have

$$
\begin{aligned}
\mathfrak{A}*\mathfrak{B} &= F(\{a_\lambda \mid \lambda \in L\} \smile \{b_\mu \mid \mu \in M\}, P, R \smile S) \\
&\gtrsim F(\{a_\lambda \mid \lambda \in L\} \smile \{b_\mu \mid \mu \in M\}, P, T) = \mathfrak{C} \supseteq \mathfrak{A}.
\end{aligned}
$$

Hence $\mathfrak{A}*\mathfrak{B}$ contains $\mathfrak{A}$. This completes the proof.

The following two corollaries can be easily obtained.

Corollary 1. *A free P-product of any P-algebraic system $\mathfrak{A}$ and a free P-algebraic system $F(\{x\}, P, \phi)$ is a P-extension of $\mathfrak{A}$.*

Corollary 2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two P-algebraic systems. If $\mathfrak{A}$ contains a one-element subsystem, then $\mathfrak{A}$ is contained in the free P-product of $\mathfrak{A}$ and $\mathfrak{B}$.*

A family P of implications $A_\kappa(x_1, \cdots, x_{r_\kappa}) \Rightarrow B_\kappa(x_1, \cdots, x_{r_\kappa})$ is said to be regular, if, for any P-algebraic system $\mathfrak{A}$, there exists a P-extension of $\mathfrak{A}$ which contains a one-element subsystem.

Theorem 2. *In order that a free P-product of any P-algebraic systems $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$ is a P-extension of all $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$, it is necessary and sufficient that the family P is regular.*

Proof. It is clear that there exists a one-element P-algebraic system $\mathfrak{E}$. If a free P-product of any P-algebraic systems $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$ is a P-extension of all $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$, then a free P-product of any P-algebraic system $\mathfrak{A}$ and the one-element P-algebraic system $\mathfrak{E}$ is a

P-extension of both $\mathfrak{A}$ and $\mathfrak{E}$, i.e. the family P is regular. This completes the proof of the necessity. Hereafter we shall prove the sufficiency. Now suppose that the family P is regular. Then any P-algebraic system $\mathfrak{A}_i$ is contained in a P-algebraic system $\mathfrak{A}_i^*$ with its one-element subsystem $\mathfrak{E}_i$. Since $\mathfrak{E}_i$ is clearly homomorphic to any P-algebraic system, it is clear from Theorem 1 that the free P-product $\mathfrak{A}_i*(\mathfrak{A}_1*\cdots*\mathfrak{A}_{i-1}*\mathfrak{A}_{i+1}*\cdots*\mathfrak{A}_n)$ is a P-extension of $\mathfrak{A}_i$. Hence the free P-product $\mathfrak{A}_1*\cdots*\mathfrak{A}_n$ is a P-extension of $\mathfrak{A}_i$, because $\mathfrak{A}_1*\cdots*\mathfrak{A}_n = \mathfrak{A}_i*(\mathfrak{A}_1*\cdots*\mathfrak{A}_{i-1}*\mathfrak{A}_{i+1}*\cdots*\mathfrak{A}_n)$. Therefore the free P-product $\mathfrak{A}_1*\cdots*\mathfrak{A}_n$ is a P-extension of all $\mathfrak{A}_1,\cdots,\mathfrak{A}_n$. This completes the proof.

References

[1] T. Fujiwara: Note on free algebraic systems, Proc. Japan Acad., **32** (1956).
[2] T. Fujiwara: Supplementary note on free algebraic systems, Proc. Japan Acad., **33** (1957).
[3] K. Shoda: Allgemeine Algebra, Osaka Math. J., **1** (1949).