145. On the Projection of Norm One in W^*-algebras

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In the present paper, we will study on the projection of norm one from any W^*-algebra onto its subalgebra. By a projection of norm one we mean a projection mapping from any Banach space onto its subspace whose norm is one. At first, we find some properties of a projection of norm one from a C^*-algebra to its C^*-subalgebra. These properties turn out to have some interesting applications to the recent theory of W^*-algebras, which we shall show in the following.

Through our discussions we denote the dual of a Banach space M and the second dual by M' and M'', respectively.

Theorem 1. Let M be a C^*-algebra with a unit and N its C^*-subalgebra. If π is a projection of norm one from M to N, then

1. π is order preserving, 2. $\pi(axb)=a\pi(x)b$ for all $a, b\in N$,

3. $\pi(x)^*\pi(x)\leq\pi(x^*x)$ for all $x\in M$.

Proof. Consider the second dual of M and N, M'' and N''. M'' is a W^*-algebra containing M as a σ-weakly dense C^*-subalgebra by Sherman's theorem (cf. [14, 15]), and N'' may be considered as a W^*-subalgebra of M'', for it is identified with the bipolar of N in M''. The second transpose of π, the extension of π to M'', is a projection of norm one from M'' to N''. Thus, it suffices to prove the theorem when M is a W^*-algebra and N a W^*-subalgebra of M. As in [5, Lemma 8] we can show that π is $*$-preserving and order preserving, which one can easily see since π is of norm one.

Next, take a projection e of N and $a\in M$, positive and $\|a\|\leq 1$. We have $e\geq eae$, whence $e\geq\pi(eae)$, so that $\pi(eae)=e\pi(eae)e$. Thus, we have $\pi(exe)=e\pi(exe)e$ for all $x\in M$. Take an element $x\in M$, $\|x\|\leq 1$. Put $\pi(ex(1-e))=x'$. Then

$$\|ex(1-e)+ne\|=\|\{ex(1-e)+ne\}\{(1-e)x^*e+ne\}\|^{1/2}$$
$$=\|ex(1-e)x^*e+n^2e\|^{1/2}\leq(1+n^2)^{1/2} \text{ for all integers } n.$$

On the other hand, if $\frac{ex'e+ex'^*e}{2}\neq 0$ we may suppose without loss of generality that this element has a positive spectrum $\lambda>0$. Then,

$$\|x'+ne\|=\|ex'e+ne+ex'(1-e)+(1-e)x'e+(1-e)x'(1-e)\|$$
$$\geq\|e(x'+nl)e\|\geq\left\|\frac{ex'e+ex'^*e}{2}+ne\right\|\geq\lambda+n \text{ for all } n.$$

Therefore, $\|x'+ne\|\geq\lambda+n>(1+n^2)^{1/2}\geq\|ex(1-e)+ne\|$ for a sufficiently large n, which is a contradiction. Thus $\dfrac{ex'e+ex'^*e}{2}=0$. A slight modification leads us to $\dfrac{iex'^*e-iex'e}{2}=0$. We get, $ex'e=0$. For $ex(1-e)+n(1-e)$ we proceed the same computation and get, $(1-e)x'(1-e)=0$.

Now suppose $(1-e)x'e\neq 0$. We have,

$$\begin{aligned}\|x'+n(1-e)x'e\|&=\|ex'(1-e)+(n+1)(1-e)x'e\|\\&=\max\{\|ex'(1-e)\|,\ (n+1)\|(1-e)x'e\|\}\\&=(n+1)\|(1-e)x'e\| \text{ for a sufficiently large } n.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|x'+n(1-e)x'e\|&\leq\|ex(1-e)+n(1-e)x'e\|\\&=\max\{\|ex(1-e)\|,\ n\|(1-e)x'e\|\}\\&=n\|(1-e)x'e\| \text{ for a sufficiently large } n.\end{aligned}$$

This is a contradiction, which yields $(1-e)x'e=0$. Thus we have $x'=ex'(1-e)$. Since $\pi(x)=\pi(exe)+\pi(ex(1-e))+\pi((1-e)xe)+\pi((1-e)\cdot x(1-e))$, we have $e\pi(x)(1-e)=e\pi(ex(1-e))(1-e)=\pi(ex(1-e))$, and $e\pi(x)e=e\pi(exe)e=\pi(exe)$. Therefore $\pi(ex)=e\pi(x)$. We have $\pi(ax)=a\pi(x)$ for all $a\in N$, because N is a W^*-subalgebra of M. Since these arguments are symmetric we get the conclusion 2°.

From 2°, 3° is easily shown: that is,

$$\begin{aligned}0\leq\pi[(x-\pi(x))^*(x-\pi(x))]&=\pi(x^*x-x^*\pi(x)-\pi(x)^*x+\pi(x)^*\pi(x))\\&=\pi(x^*x)-\pi(x)^*\pi(x).\end{aligned}$$

By help of Theorem 1 we can prove the following theorem on W^*-algebra which is proved recently by S. Sakai [12].

Theorem 2. Suppose a C^*-algebra M is the adjoint space of a Banach space F, then it is a W^*-algebra and the topology $\sigma(M,F)$ of M is the σ-weak topology.

Proof. By [2] there exists a projection π of norm one from M'' to M whose kernel is the polar of F in M''. Then, by Theorem 1, $a\,\pi^{-1}(0)b\subset\pi^{-1}(0)$ for all $a,b\in M$. Since M is a σ-weakly dense C^*-subalgebra of M'', we have

$$x\pi^{-1}(0)y\subset\pi^{-1}(0)\quad\text{for all } x,y\in M''.$$

Thus $\pi^{-1}(0)$ is a σ-weakly closed ideal of M'' and π is a $*$-homomorphism from M'' onto M. Therefore M is isomorphic to $M''/\pi^{-1}(0)$ which is a W^*-algebra, that is, M is a W^*-algebra. The σ-weak topology of a W^*-algebra $M''/\pi^{-1}(0)$ is the quotient topology of the σ-weak topology of M'' which is equivalent to $\sigma(M'',M')$-topology (cf. [15]). Therefore the $\sigma(M,F)$-topology of M is the σ-weak topology of M by [1].

Combining this result with that of J. Dixmier [2] we get

Corollary. A C^*-algebra M is a W^*-algebra if and only if there exists a projection of norm one from M'', the second dual of M, to

M whose kernel is $\sigma(M'', M')$-closed.

Next, we apply this method to the following

Theorem 3 (*cf.* [*13, Theorem 2*]). *Let M be a W^*-algebra, N a C^*-algebra and ϕ an algebraic isomorphism from M onto N, then N is a W^*-algebra and is σ-weakly bicontinuous.*

Proof. By [11] ϕ is uniformly continuous, so that it is bicontinuous by the classical theorem of Banach space. Let M'' and N'' be the second duals of M and N, then ϕ induces a σ-weakly bicontinuous isomorphism between two W^*-algebras M'' and N'' which is nothing but the second transpose of ϕ, $\overset{\approx}{\phi}$. Since M is a W^*-algebra, there exists a projection π_0 of norm one described in the previous corollary. Put $\pi_1=\phi\pi_0\overset{\approx}{\phi}{}^{-1}$: π_1 is a projection from N'' to N and $\pi_1^{-1}(0)=\overset{\approx}{\phi}\pi_0^{-1}(0)$. Therefore $\pi_1^{-1}(0)$ is $\sigma(N'', N')$-closed. Moreover $\pi_1^{-1}(0)$ is an ideal since $\pi_0^{-1}(0)$ is an ideal of M'' as it is seen in the proof of Theorem 2. Hence N is $*$-isomorphic to a W^*-algebra $N''/\pi_1^{-1}(0)$, so that N is a W^*-algebra. Now let $\pi_1^{-1}(0)^0$ be the polar of $\pi_1^{-1}(0)$ in N', then $\pi_1^{-1}(0)^0$ may be regarded as N_*, the space of all σ-weakly continuous linear functionals on N, by Theorem 2. Denote the polar of $\pi_0^{-1}(0)$ in M' by $\pi_0^{-1}(0)^0$, we have $\pi_0^{-1}(0)^0=M_*$. Then

$$\langle\tilde{\phi}(\pi_1^{-1}(0)^0),\ \pi_0^{-1}(0)\rangle=\langle\pi_1^{-1}(0)^0,\ \overset{\approx}{\phi}\pi_0^{-1}(0)\rangle=\langle\pi_1^{-1}(0)^0,\ \pi_1^{-1}(0)\rangle=0, \text{ and}$$

$$\langle\tilde{\phi}^{-1}(\pi_0^{-1}(0)^0),\ \pi_1^{-1}(0)\rangle=\langle\pi_0^{-1}(0)^0,\ \tilde{\phi}^{-1}\pi_1^{-1}(0)\rangle=\langle\pi_0^{-1}(0)^0,\ \overset{\approx}{\phi}{}^{-1}\pi_1^{-1}(0)\rangle$$
$$=\langle\pi_0^{-1}(0)^0,\ \pi_0^{-1}(0)\rangle=0.$$

Therefore ϕ is σ-weakly bicontinuous.

Theorem 4. Let M be a W^*-algebra, N a C^*-subalgebra of M and π a projection of norm one from M to N, then

1°. *N is a W^*-algebra if $\pi^{-1}(0)\frown\bar{N}$ is σ-weakly closed where $\bar{N}$ is the σ-weak closure of N in M,*

2°. *N is a W^*-subalgebra if π is faithful on positive elements in M.*

Proof. Since $\pi(\bar{N})=N$, it suffices to consider the restriction of π to N. By Corollary of Theorem 2 there exists a projection π_0 of norm one from N'' to N. Consider the restriction of π_0 to N'' which is a W^*-subalgebra of N'' as shown in the proof of Theorem 1. By the proof of Theorem 2, we see that π_0 is a σ-weakly continuous $*$-homomorphism of $\bar{N}''$ onto N, so that $\pi_0(N'')$ is σ-weakly closed in $\bar{N}$ containing N (cf. [4]). Hence $\pi(N'')=\bar{N}$. Put $\pi_1=\pi\pi_0$ on N'', then π_1 is a projection of norm one from N'' to N: moreover, $\pi_1^{-1}(0)=\pi_0^{-1}(\pi^{-1}(0)\frown\bar{N})\frown N''$, which is σ-weakly closed by the σ-weak topology in N'', that is, $\sigma(N'', N')$-topology. Therefore N is a W^*-algebra, which proves 1°.

Next, if $\{a_\alpha\}$ is a bounded increasing directed set of self-adjoint elements of N, there exists an element a_0 in M such that $a_0 = \sup_\alpha a_\alpha$. Since π is order preserving, a simple computation shows $\pi(a_0) = \sup_\alpha a_\alpha$ in N. Hence, we have $\pi(a_0) \geq a_0$, that is, $\pi(a_0) - a_0 \geq 0$. Then, $\pi(\pi(a_0) - a_0) = 0$ which implies $\pi(a_0) - a_0 = 0$ since π is faithful on positive elements. Therefore N is a C^*-algebra in which the supremum of each bounded increasing directed set in N coincides with that in a W^*-algebra M. Hence N is a W^*-subalgebra of M owing to the result due to Kadison [6]. This proves 2°.

Remark. It is to be noticed that the first half part of Theorem 4 does not necessarily hold without any additional assumption. For example, take a commutative AW^*-algebra N whose spectrum space is not a hyperstonean space. N is a C^*-algebra on a Hilbert space H. Let M be the σ-weak closure of N on H. M is a commutative W^*-algebra. Denote the self-adjoint parts of M and N by M_s and N_s, respectively. By [9, 10] there exists a projection of norm one from M_s onto N_s. Then we can extend this projection linearly to a projection from M to N without increasing its norm. Thus, we have a projection of norm one from M onto N and yet N is not a W^*-algebra (cf. [3]).

In the case of AW^*-algebra, we have

Theorem 5. Let M be an AW^*-algebra, N its C^*-subalgebra and π a projection of norm one from M to N, then

1°. *N is an AW^*-algebra,*

2°. *N is an AW^*-subalgebra if π is faithful on positive elements in M.*

Proof. Let S be an arbitrary set in N and denote by R_0 and R the right annihilator in M and N, respectively. We have $R_0 = eM$ for some projection e. Now, by Theorem 1, $Se = 0$ implies $\pi(Se) = S\pi(e) = 0$. Hence there exists an element $a \in M$ such that $\pi(e) = ea$. We get, therefore,

$$\pi(e)^2 = \pi(e)\pi(e) = \pi(e\pi(e)) = \pi(\pi(e)) = \pi(e),$$

so that $\pi(e)$ is a projection in N for $\pi(e)$ is positive. Besides, we have $\pi(e)N \subset R$. On the other hand, $\pi(e)N \supset \pi(e)R = \pi(eR) = \pi(R) = R$. We get $R = \pi(e)N$. That is, N is an AW^*-algebra (cf. [8]).

To prove the second half of the theorem, we consider (e_α), a family of orthogonal projections in N. Since N is an AW^*-algebra by 1°, there exists a projection e in N such that $e = \sup_\alpha e_\alpha$ in N. On the other hand we have a projection e_0 in M such that $e_0 = \sup_\alpha e_\alpha$ in M. And the same computation as in the proof of 2° in Theorem 4 shows that $\pi(e_0) = e = e_0$ if π is faithful on positive elements in M. Thus N is an AW^*-subalgebra of M (cf. [7]).

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