140. On Eigenfunction Expansions of Self-adjoint Ordinary Differential Operators. I

By Takashi KASUGA

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In this note, we shall prove some results about eigenfunction expansions of self-adjoint ordinary differential operators for the case when one of their characteristic functions[1] is meromorphic on some parts of the real line R.

§1. Let us consider the differential expression

$L[u]=-(d/dx)\{p(x)d/dx\}u+q(x)\cdot u \quad (a<x<b, -\infty\leqq a<b\leqq+\infty)$

defined on a (finite or infinite) open interval (a,b), where $p(x)$, $q(x)$ are real-valued functions defined in (a,b), $p(x)$ has continuous first derivative, $q(x)$ is continuous, and $p(x)>0$ for $a<x<b$.

Following H. Weyl,[2] we classify L according to its behaviour in the neighbourhood of the point a (or b), in *the l. c. type* (*limit circle type*) at a (or b) and *the l. p. type* (*limit point type*) at a (or b).

In this note, all functions are complex-valued if not specially noted.

$\mathfrak{H}_I$: the set of functions defined on (a,b) and square summable on I, where I is a subinterval (open, closed, or half-open) of (a,b).

$\mathfrak{H}$: the set $\mathfrak{H}_{(a,b)}$ of functions.

$\mathfrak{D}$: the set of functions u defined on (a,b) such that u is differentiable on (a,b) and du/dx is absolutely continuous on every finite closed subinterval of (a,b).

$\mathfrak{G}_a$ (or $\mathfrak{G}_b$): the set of functions belonging to $\mathfrak{D}$ such that u, $L[u]\in\mathfrak{H}_{(a,c]}$ (or $\mathfrak{H}_{[c,b)}$) for every point c of (a,b).

Bracket. For $u,v\in\mathfrak{D}$, we introduce the bracket:

$$[uv](x)=p(x)[u(x)v'(x)-v(x)u'(x)]$$
$$(u'=du/dx,\ v'=dv/dx).$$

In case u and v satisfy one and the same equation $L[u]=l\cdot u$, we write $[uv]$ for $[uv](x)$, since, in this case $[uv](x)$ does not depend on x.[3]

If L is of the l. c. type at a (or b) and $u,w\in\mathfrak{G}_a$ (or $\mathfrak{G}_b$), the limit $[wu](a)=\lim_{x\to a}[wu](x)$ (or $[wu](b)=\lim_{x\to b}[wu](x)$) exists.[4]

Boundary conditions.

$\mathfrak{G}'_a$ (or $\mathfrak{G}'_b$): the set $\mathfrak{G}_a$ (or $\mathfrak{G}_b$) of functions if $L[u]$ is of the l. p.

1) Cf. §1.
2) Cf. Weyl [7], Titchmarsh [6], Coddington and Levinson [2].
3) Cf. the reference quoted in 2).
4) Cf. the reference quoted in 2).

type at a (or b), and if $L[u]$ is of the l.c. type at a (or b), then the set of functions u belonging to $\mathfrak{G}_a$ (or $\mathfrak{G}_b$) satisfying a non-trivial[5] condition

$$[w_a u](a)=0 \text{ (or } [w_b u](b)=0)$$

where w_a (or w_b) belongs to $\mathfrak{G}_a$ (or $\mathfrak{G}_b$), is real and is fixed once for all in the following.

The differential operator.
If we put

$$Hu=L[u] \text{ for } u\in\mathfrak{G}'_a\cap\mathfrak{G}'_b,$$

then we obtain a self-adjoint operator H in $\mathfrak{H}$.[6]

Fundamental solutions. By a system of fundamental solutions of $L[u]$, we shall mean the system of two solutions $s_1(x, l)$, $s_2(x, l)$ in $\mathfrak{D}$ of $L[u]=l\cdot u$ such that

i) $[s_2 s_1]=1$

ii) $s_k(x, \bar{l})=\overline{s_k(x, l)}$[7] $k=1, 2$

iii) as functions of l, $s_k(x, l)$ and $(d/dx)s_k(x, l)$ $(k=1, 2)$ are regular analytic in the whole complex l-plane.

$s_k(x, l)$, $(d/dx)s_k(x, l)$ $(k=1, 2)$ are continuous as functions of (x, l) for x belonging to (a, b) and l on the whole complex plane.

For two complex numbers l, f, we define

$\mathfrak{F}(l, f)$: the family of functions defined for $a<x<b$ of the form

$$C[s_2(x, l)+f\cdot s_1(x, l)]$$

where C is an arbitrary complex number.

$\mathfrak{F}(l, \infty)$: the family of functions defined for $a<x<b$ of the form

$$Cs_1(x, l)$$

where C is an arbitrary complex number.

Characteristic functions. For a complex number l such that $\mathfrak{J}l\neq 0$,[8] there is a uniquely determined point $f_a(l)$ (or $f_b(l)$) of Riemann sphere (the complex plane augmented by the infinity) such that

$$\mathfrak{F}\{l, f_a(l)\}\subset\mathfrak{G}'_a \text{ (or } \mathfrak{F}\{l, f_b(l)\}\subset\mathfrak{G}'_b).^{9)}$$

We call $f_a(l)$, $f_b(l)$, defined for $\mathfrak{J}l\neq 0$, *the characteristic functions of H.*

$f_a(l)$ and $f_b(l)$ are meromorphic on the upper and the lower half complex planes ($\mathfrak{J}l\neq 0$) and

$$f_a(\bar{l})=\overline{f_a(l)} \quad f_b(\bar{l})=\overline{f_b(l)} \quad f_a(l)\neq f_b(l) \text{ for } \mathfrak{J}l\neq 0.^{10)} \tag{1}$$

If $L[u]$ is of the l.c. type at a (or b), then $f_a(l)$ (or $f_b(l)$) is meromorphic and

5) A condition for $u\in\mathfrak{G}_a$ (or $\mathfrak{G}_b$) of the form $[w_a u](a)=0$ (or $[w_b u](b)=0$) where $w_a\in\mathfrak{G}_a$ (or $w_b\in\mathfrak{G}_b$), is called trivial if $[w_a u]=0$ (or $[w_b u]=0$) for all $u\in\mathfrak{G}_a$ (or $\mathfrak{G}_b$).

6) Cf. Stone [5], Weyl [7, 8].

7) The bar means the conjugate complex number.

8) In the following, $\mathfrak{J}l$ and $\mathfrak{R}l$ mean the imaginary part and the real part of l respectively.

9) Cf. the reference quoted in 2).

10) Cf. Weyl [9].

$$\mathfrak{F}\{l, f_a(l)\}\subset\mathfrak{G}_a' \text{ (or } \mathfrak{F}\{l, f_b(l)\}\subset\mathfrak{G}_b')$$

on the whole complex l-plane.[11]

Change of system of fundamental solutions. When we put

$$\tilde{s}_j(x, l)=\sum_{k=1,2}\beta_{jk}(l)s_k(x, l) \quad (j=1, 2) \tag{2}$$

for a system of fundamental solutions $s_k(x, l)$ $(k=1, 2)$ of $L[u]=l\cdot u$, $\tilde{s}_j(x, l)$ $(j=1, 2)$ constitute another system of fundamental solutions of $L[u]=l\cdot u$, if and only if $\beta_{jk}(l)$ $(j, k=1, 2)$ are (transcendental) entire functions of l and

$$\beta_{jk}(\bar{l})=\overline{\beta_{jk}(l)} \quad (j, k=1, 2) \quad \det(\beta_{jk}(l))=1. \tag{3}$$

Also when we denote f_a, f_b, $\mathfrak{F}(l, f)$ corresponding to the new system of fundamental solutions $\tilde{s}_1$, $\tilde{s}_2$ by $\tilde{f}_a$, $\tilde{f}_b$, $\tilde{\mathfrak{F}}(l, \tilde{f})$, then

$$\tilde{\mathfrak{F}}(l, \tilde{f})=\mathfrak{F}(l, f) \tag{4}$$

if and only if

$$f=\{\beta_{21}(l)+\beta_{11}(l)\tilde{f}\}/\{\beta_{22}(l)+\beta_{12}(l)\tilde{f}\}. \tag{5}$$

On the other hand,

$$\mathfrak{F}\{l, f_b(l)\}=\tilde{\mathfrak{F}}\{l, \tilde{f}_b(l)\} \quad (\text{or } \mathfrak{F}\{l, f_a(l)\}=\tilde{\mathfrak{F}}\{l, \tilde{f}_a(l)\})$$

for $\Im l\neq 0$, by the definition of $f_b(l)$ (or $f_a(l)$).

Hence we get for $\Im l \neq 0$

$$f_b(l)=\{\beta_{21}(l)+\beta_{11}(l)\tilde{f}_b(l)\}/\{\beta_{22}(l)+\beta_{12}(l)\tilde{f}_b(l)\} \tag{6}$$

and a similar formula for $f_a(l)$.

By (6), (5), (4), we can easily prove:

Lemma 1. If $f_b(l)$ (or $f_a(l)$) is meromorphic in a neighbourhood of a real l_0, then also $\tilde{f}_b(l)$ (or $\tilde{f}_a(l)$) is meromorphic in a neighbourhood of l_0 and

$$\mathfrak{F}\{l_0, f_b(l_0)\}=\tilde{\mathfrak{F}}\{l_0, \tilde{f}_b(l_0)\} \quad (\text{or } \mathfrak{F}\{l_0, f_a(l_0)\}=\tilde{\mathfrak{F}}\{l_0, \tilde{f}_a(l_0)\}).$$

Also we remark here that if $f_b(l)$ (or $f_a(l)$) is regular in a neighbourhood of a real l_0, $f_b(l_0)$ (or $f_a(l_0)$) is real by (1).

§2. Theorem 1. If $f_b(l)$ (or $f_a(l)$) is meromorphic in a neighbourhood of a real l_0, then

$$\mathfrak{F}\{l_0, f_b(l_0)\}\subset\mathfrak{G}_b' \quad (\text{or } \mathfrak{F}\{l_0, f_a(l_0)\}\subset\mathfrak{G}_a').$$

Proof. We shall prove the theorem for the end point b. The proof of the theorem for the end point a goes quite similarly.

If $L[u]$ is of the l. c. type at b, then the theorem is already obvious from the propositions stated at the end of the definition of the characteristic functions in §1. Hence we assume that $L[u]$ is of the l. p. type at b.

By Lemma 1, the premise and the conclusion of the theorem have invariant meanings under the change of the system of fundamental solutions $s_1(x, l)$, $s_2(x, l)$. Hence for the proof of the theorem we take a special system of fundamental solutions

11) Cf. Weyl [9].

$$S(c', \theta)=\{s_1(x, l),\ s_2(x, l)\}$$

which satisfy the conditions

$$s_2(c', l)=\sin\theta \qquad s_1(c', l)=\cos\theta$$
$$p(c')s_2'(c, l)=-\cos\theta \quad p(c')s_1'(c', l)=\sin\theta$$

for a point c' of (a, b) and a real θ.

If we write $S(c', \theta-\pi/2)=\{\tilde{s}_1(x, l), \tilde{s}_2(x, l)\}$, then $\tilde{s}_1(x, l)=s_2(x, l)$ $\tilde{s}_2(x, l)=-s_1(x, l)$.

In this case, (6) becomes

$$f_b(l)=-\tilde{f}_b^{-1}(l)$$

so that

$$\tilde{f}_b(l)=-f_b^{-1}(l).$$

Therefore if $f_b(l)$ has a pole at l_0, then $\tilde{f}_b(l)$ is regular in the neighbourhood of l_0. Hence we can assume that $f_b(l)$ is regular in the neighbourhood of l_0, by taking the system $S(c', \theta-\pi/2)$ in the place of $S(c', \theta)$, if necessary.

For the special system $S(c', \theta)=\{s_1(x, l), s_2(x, l)\}$, we have[12]

$$\int_{c'}^{r} |s_2(x, l)+f_b(l)s_1(x, l)|^2\,dx<\Im f_b(l)\varepsilon^{-1} \tag{7}$$

for $c'<r<b$ and $l=l_0+i\varepsilon$ $(\varepsilon>0)$.

If $f_b(l)$ is regular in the neighbourhood of l_0, then

$$\Im f_b(l_0+i\varepsilon)=O(\varepsilon) \quad \text{for } \varepsilon\to+0$$

since $f_b(l)$ is real on the real line in the neighbourhood of l_0. Hence by (7)

$$\int_{c'}^{r}|s_2(x, l_0+i\varepsilon)+f_b(l_0+i\varepsilon)s_1(x, l_0+i\varepsilon)|^2dx<\Im f_b(l_0+i\varepsilon)\varepsilon^{-1}<M\,(>0) \tag{8}$$

where M does not depend on r $(c'<r<b)$ and ε $(0<\varepsilon<\varepsilon_0)$. On the other hand, $s_2(x, l_0+i\varepsilon)+f_b(l_0+i\varepsilon)s_1(x, l_0+i\varepsilon)\to s_2(x, l_0)+f_b(l_0)s_1(x, l_0)$ uniformly in the interval $c'\leqq x\leqq r$ for $\varepsilon\to+0$ since $s_1(x, l)$, $s_2(x, l)$ are continuous as functions of (x, l) on their whole domain of definition and $f_b(l)$ is regular in the neighbourhood of l_0.

Therefore letting $\varepsilon\to+0$ in (8), we have

$$\int_{c'}^{r} |s_2(x, l_0)+f_b(l_0)s_1(x, l_0)|^2\,dx\leqq M. \tag{9}$$

Letting $r\to b$ in (9), we get

$$\int_{c'}^{b} |s_2(x, l_0)+f_b(l_0)s_1(x, l_0)|^2\,dx\leqq M<+\infty.$$

But $s_2(x, l_0)+f(l_0)s_1(x, l_0)$ is square summable on every finite closed subinterval of (a, b). Hence $s_2(x, l_0)+f_b(l_0)s_1(x, l_0)\in\mathfrak{H}_{[c,b)}$ for every point c of (a, b). Also from this and the fact that $s_2(x, l_0)+f_b(l_0)s_1(x, l_0)$ is a solution of $L[u]=l_0\cdot u$, it follows that $s_2(x, l_0)+f_b(l_0)s_1(x, l_0)\in\mathfrak{D}$ and

12) Cf. Coddington and Levinson [2, p. 228].

$L[s_2(x, l_0)+f_b(l_0)s_1(x, l_0)] \in \mathfrak{H}_{[a,b)}$ for any $c \in (a, b)$. This concludes the proof of Theorem 1.

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