31. Some Transformation Equations in the Theory of Partitions

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In a recent paper [2] the author has obtained the following functional equation:

$$\sum_{l=0}^{\infty}\{\lambda((l+\alpha)z-i\beta)+\lambda((l+1-\alpha)z+i\beta)\}+\pi z(\alpha^2-\alpha+1/6)$$
$$(1)\qquad =\sum_{l=0}^{\infty}\{\lambda((l+\beta)/z+i\alpha)+\lambda((l+1-\beta)/z-i\alpha)\}+\frac{\pi}{z}(\beta^2-\beta+1/6)$$
$$+2\pi i(\alpha-1/2)(\beta-1/2),$$

where $0\leqq\alpha\leqq1$, $0<\beta<1$ (or $0<\alpha<1$, $0\leqq\beta\leqq1$), z is a complex number with $\Re(z)>0$, and $\lambda(t)$ denotes $-\log(1-e^{-2\pi t})$. This formula may be expected to have some applications in the theory of partitions. Indeed, the famous transformation formula for the Dedekind modular function $\eta(\tau)$ can be easily derived from (1) (see [2]).

In the present paper we shall note that formula (1) will also yield a transformation equation of the generating function for $p(n; a, M)$, the number of partitions of a positive integer n into positive summands of the form $Ml\pm a$ $(l=0, 1, 2,\cdots)$, where a and M are integers such that $M\geqq2$, $0<a<M$, $(a, M)=1$.

This partition function has been treated for special values of M by several writers; namely, the case $M=2$ by Hua [1], the case $M=6$ by Niven [5], the case $M=5$ by Lehner [3] and generally the case $M=p$, where p is a prime greater than 3, by Livingood [4]; each resulting in a convergent series expansion for the partition function, by using the Farey-dissection method first introduced by Hardy and Ramanujan and later improved by Kloosterman and Rademacher.

In the use of the Farey-dissection method it is important to find the transformation equation to determine the asymptotic behavior of the generating function near its singularity at each 'rational point' on the unit circle.

The generating function of $p(n; a, M)$ is clearly found to be, for $M\geqq3$,

$$F(x; a, M)=1+\sum_{n=1}^{\infty}p(n; a, M)\ x^n=\prod_{l=0}^{\infty}(1-x^{Ml+a})^{-1}(1-x^{Ml+M-a})^{-1}\qquad(|x|<1).$$

But for $M=2$, i.e. for partitions into odd parts (or equivalently unequal parts), we have

(2) $p(n; 1, 2)=p(n; 1, 4), \quad F(x; 1, 2)=\prod_{l=0}^{\infty}(1-x^{2l+1})^{-1}=F(x; 1, 4).$

Therefore it suffices to consider the cases $M\geqq 3$ in what follows.

The desired transformation equation for $F(x;a,M)$ will be obtained, by some elementary algebraic manipulation, from formula (1), and it is unnecessary to make any use of the theory of elliptic modular functions.

Our main theorem may be stated as follows:

Theorem 1.[1] *Let z be a complex number with $\Re(z)>0$, and h, k be coprime integers with $k\geqq 1$. Denote by D and K the* g.c.m. *and the* l.c.m. *of k and M, respectively. Put $k=k_1D$, $M=m_1D$, so that $(k_1, m_1)=1$, and choose any integers γ, δ satisfying $\gamma k_1-\delta m_1=1$. Let, then, H be any solution of the congruence $hH\equiv\delta \pmod k$. Further, set*

$$x=\exp(2\pi ih/k-2\pi z/k), \quad \tilde{x}=\exp(2\pi iH/k-2\pi/Kz),$$

$$F(\tilde{x}; b, D, \rho)=\prod_{l=0}^{\infty}(1-\rho\tilde{x}^{Dl+b})^{-1}(1-\bar{\rho}\tilde{x}^{Dl+D-b})^{-1},$$

where

$$b=ha-D[ha/D], \quad \rho=\exp(-2\pi ia\gamma/M), \quad \bar{\rho}=\exp(2\pi ia\gamma/M),$$

$[t]$ denoting the greatest integer not exceeding t.

Then, if $M\geqq 3$, we have the equation

(3) $F(x; a, M)=\omega(h, k)\exp\{(\pi/6kM)(B/z-Az)\}F(\tilde{x}; b, D, \rho)$

with the notations

$$\omega(h, k)=\exp\{2\pi i\sigma(h, k)\}, \quad \sigma(h, k)=\sum_{\mu}(\mu/K-1/2)(h\mu/k-[h\mu/k]-1/2),$$

where μ runs over the integers $a, a+M, a+2M, \cdots, a+(k_1-1)M$; and

$$A=6a^2-6Ma+M^2, \quad B=6b^2-6Db+D^2.$$

The case $D=1$ of Theorem 1 can also be expressed in a different form.

Theorem 2. *With the same notations as in Theorem 1, we have, when $(k, M)=1$, $M\geqq 3$, the following equation:*

(4) $$F(x; a, M)=\tfrac{1}{2}\chi(h, k)\csc(\pi\xi/M)\exp\{(\pi/6kM)(1/z-Az)\} \times\prod_{l=1}^{\infty}(1-\rho\tilde{x}^{l})^{-1}(1-\bar{\rho}\tilde{x}^{l})^{-1},$$

where ξ is an integer defined by $\xi k\equiv a \pmod M$ $(0<\xi<M)$; and

$$\chi(h, k)=\exp\{2\pi i\tau(h, k)\}, \quad \tau(h, k)=\sum_{\mu}((\mu/Mk))((h\mu/k))$$

with the abbreviation

$$((t))=\begin{cases}0, & \text{if } t \text{ is an integer,}\\ t-[t]-1/2, & \text{otherwise.}\end{cases}$$

Further we have $\rho=\exp(-2\pi i\xi/M)$, and H is also determined by the congruence $MhH\equiv -1 \pmod k$.[2]

1) A detailed proof of Theorem 1 will be published in a future paper.

2) This congruence was employed for the definition of H by Lehner [3] and Livingood [4], where they wrote H' for our H and H for our Mh.

Proof. First, since $(h, k)=1$, $h\mu/k$ is integer if and only if $k|\mu$, that is, if and only if $\mu=\xi k$ as is seen from $\mu\equiv a \pmod{M}$ and $0<\mu<Mk$. Thus we get, noting that $K=Mk$,

$$\sigma(h, k)=\tau(h, k)+(\xi k/K-1/2)(-1/2)=\tau(h, k)+(1/2-\xi/M)/2.$$

On the other hand, $D=1$ implies $b=0$ and also $B=1$, as is obvious from their definitions in Theorem 1. Hence formula (3) is written in the form

$$(5)\quad \begin{aligned} F(x; a, M)=i \exp(-\pi i\xi/M)\chi(h, k) \exp\{(\pi/6kM)(1/z-Az)\} \\ \times(1-\rho)^{-1}\prod_{l=1}^{\infty}(1-\rho\tilde{x}^{l})^{-1}(1-\bar{\rho}\tilde{x}^{l})^{-1}. \end{aligned}$$

Moreover, since $\gamma k\equiv 1 \pmod{M}$, we have $a\gamma k\equiv a \pmod{M}$, which yields $a\gamma k\equiv \xi k \pmod{M}$, so that $a\gamma\equiv\xi \pmod{M}$ by virtue of $(k, M)=1$. Hence

$$\rho=\exp(-2\pi i a\gamma/M)=\exp(-2\pi i\xi/M),$$

and also

$$i \exp(-\pi i\xi/M)(1-\rho)^{-1}=i \exp(-\pi i\xi/M)(2i)^{-1}\exp(\pi i\xi/M)\csc(\pi\xi/M) \\ =\tfrac{1}{2}\csc(\pi\xi/M).$$

Thus (4) follows immediately from (5).

Finally, the congruence $hH\equiv\delta \pmod{k}$ is equivalent to $MhH\equiv-1 \pmod{k}$ since $\delta M\equiv-1 \pmod{k}$.

This completes the proof of Theorem 2.

We remark that $h\mu/k$ is always non-integral for the case $D>1$, since the congruence $\xi k\equiv a \pmod{M}$ has no solutions in ξ inasmuch as we have assumed $(a, M)=1$ above. Hence, observing that $0<\mu<K$, we may write $\sigma(h, k)=\sum_{\mu}((\mu/K))((h\mu/k))$ for $D>1$.

We shall now discuss the special case $a=1$, $M=4$. This case is of particular interest since it is equivalent to the case $M=2$, as has been shown in (2).

We consider three cases separately according as $(k, 4)=4$ or 2 or 1.

Case (i): $(k, 4)=4$. With the notations in Theorem 1, we have $D=4$, $K=k$, $k_1=k/4$, $m_1=1$; $\gamma k/4-\delta=1$, and we may choose $\gamma=0$, $\delta=-1$, so that the congruence for H becomes $hH\equiv-1 \pmod{k}$. We get further

$$\tilde{x}=\exp(2\pi iH/k-2\pi/kz),\quad b=1 \text{ or } 3,\quad \rho=\bar{\rho}=1,$$

$$F(\tilde{x}; b, 4, 1)=\prod_{l=0}^{\infty}(1-\tilde{x}^{4l+1})^{-1}(1-\tilde{x}^{4l+3})^{-1}=F(\tilde{x}; 1, 4);$$

$$\sigma(h, k)=\sum_{\mu}((\mu/k))((h\mu/k))\quad (\mu=1, 5, 9, \cdots, k-3);$$

and

$$A=B=-2.$$

Case (ii): $(k, 4)=2$. We have $D=2$, $K=2k$, $k_1=k/2$, $m_1=2$; $\gamma k/2-2\delta=1$, and we may choose $\gamma=1$, $\delta=(k/2-1)/2$. Then

$$\tilde{x}=\exp(2\pi iH/k-\pi/kz),\quad hH\equiv(k/2-1)/2 \pmod{k},$$

$$b=1,\quad \rho=\exp(-\pi i/2)=-i,\quad \bar{\rho}=i,$$

$$F(\tilde{x}; 1, 2, -i) = \prod_{l=0}^{\infty} (1+i\tilde{x}^{2l+1})^{-1}(1-i\tilde{x}^{2l+1})^{-1}$$
$$= \prod_{l=0}^{\infty} (1-(-\tilde{x}^2)^{2l+1})^{-1} = F(-\tilde{x}^2; 1, 4);$$
$$\sigma(h, k) = \sum_{\mu} ((\mu/2k))((h\mu/k)) \quad (\mu=1, 5, 9, \cdots, 2k-3);$$

and

$$A = B = -2.$$

Now let H' be any solution of $hH' \equiv -1 \pmod{k}$. Then since

$$h(2H+k/2) \equiv k/2-1+hk/2 = -1+k(h+1)/2 \equiv -1 \pmod{k},$$

we have $H' \equiv 2H+k/2 \pmod{k}$. Therefore $-\tilde{x}^2 = \exp(2\pi iH'/k - 2\pi/kz)$.

Case (iii): $(k, 4)=1$. By Theorem 2, we have

$$F(x; 1, 4) = \tfrac{1}{2}\chi(h, k) \csc(\pi\xi/4) \exp\{(\pi/24k)(1/z+2z)\} \times \prod_{l=1}^{\infty} (1-\rho\tilde{x}^l)^{-1}(1-\bar{\rho}\tilde{x}^l)^{-1}.$$

Here ξ is defined by $\xi k \equiv 1 \pmod{4}$ $(0<\xi<4)$, which yields

$$\xi = \begin{cases} 1, & \text{if } k \equiv 1 \pmod{4}, \\ 3, & \text{if } k \equiv 3 \pmod{4}. \end{cases}$$

Hence

$$\rho = \exp(-\pi i\xi/2) = \begin{cases} -i, & \text{if } k \equiv 1 \pmod{4}, \\ i, & \text{if } k \equiv 3 \pmod{4}, \end{cases}$$

and

$$\csc(\pi\xi/4) = \sqrt{2} \qquad \text{for } \xi = 1, 3.$$

Further

$$\prod_{l=1}^{\infty} (1-\rho\tilde{x}^l)^{-1}(1-\bar{\rho}\tilde{x}^l)^{-1} = \prod_{l=1}^{\infty} (1+\tilde{x}^{2l})^{-1}$$
$$= \prod_{m=0}^{\infty} (1-(\tilde{x}^2)^{2m+1}) = 1/F(\tilde{x}^2; 1, 4),$$

where $\tilde{x} = \exp(2\pi iH/k - \pi/2kz)$, and H is determined by $4hH \equiv -1 \pmod{k}$. Now let H' be any solution of $2hH' \equiv -1 \pmod{k}$. Then $H' \equiv 2H \pmod{k}$, and we have $\tilde{x}^2 = \exp(2\pi iH'/k - \pi/kz)$. Moreover, $\tau(h, k) = \sum_{\mu} ((\mu/4k))((h\mu/k))$ $(\mu = 1, 5, 9, \cdots, 4k-3)$.

The above discussion establishes the following

Theorem 3. *The generating function $F(x)$ for the number of partitions into odd (or unequal) parts satisfies the equation*

$$F(x) = \omega(h, k) \exp\{(\pi/12k)(z-1/z)\}F(\tilde{x}) \tag{6}$$

for k even, where

$$x = \exp(2\pi ih/k - 2\pi z/k), \quad \tilde{x} = \exp(2\pi iH/k - 2\pi/kz),$$
$$hH \equiv -1 \pmod{k}, \quad \omega(h, k) = \exp\{2\pi i\sigma(h, k)\},$$
$$\sigma(h, k) = \begin{cases} \sum_{\mu} ((\mu/k))((h\mu/k)) \quad (\mu=1, 5, 9, \cdots, k-3), & \text{if } 4 \mid k, \\ \sum_{\mu} ((\mu/2k))((h\mu/k)) \quad (\mu=1, 5, 9, \cdots, 2k-3), & \text{if } 4 \nmid k; \end{cases}$$

and the equation

$$F(x) = \frac{1}{\sqrt{2}}\chi(h, k) \exp\{(\pi/12k)(z+1/2z)\}/F(x') \tag{7}$$

for k odd, where

$$x=\exp(2\pi ih/k-2\pi z/k), \quad x'=\exp(2\pi iH'/k-\pi/kz),$$
$$2hH'\equiv -1 \pmod{k}, \quad \chi(h, k)=\exp\{2\pi i\tau(h, k)\},$$
$$\tau(h, k)=\sum_{\mu}((\mu/4k))((h\mu/k)) \quad (\mu=1, 5, 9, \cdots, 4k-3).$$

It is easily verified that the transformation equations (6), (7) are essentially the same ones as in Hua's paper [1], apart from the formulas for $\sigma(h, k)$ and $\tau(h, k)$, which are complicated in Hua's results, whereas in our case they are defined by analogues of Dedekind sums.

In closing this note we mention that a convergent series representation for $p(n; a, M)$, where M assumes general values, may be obtained by using the Farey-dissection method and applying Theorem 1; it is hoped to publish a full account of the result in due course.

References

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