30. On a Generalization of the Concept of Functions

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1. L. Schwartz has generalized the concept of functions on a C^∞-manifold by introducing his notion of *distributions*, which revealed to be most useful in various branches of analysis.[1] We now propose to introduce another generalization of the function concept in case the underlying manifold is C^ω (instead of C^∞) in utilizing the boundary values of analytic functions. This new concept will comprise that of Schwartz's distributions in case of C^ω-manifold. We shall call a *hyperfunction* a "function" in this generalized sense, defined precisely as follows. (For brevity, we give here the definition of the hyerfunction only in the 1-dimensional case, though we can define it for n-dimensional C^ω-manifolds.)[2]

Let R be the real axis $(-\infty, \infty)$ which we shall consider as lying in the complex plane $\boldsymbol{C}$. Let N be a locally compact subset of R. The family of all the "complex nbds of N", i.e. the open sets $D, D_1, D_2, \cdots$ of $\boldsymbol{C}$ which contain N as a closed subset, will be denoted by $\mathfrak{D}(N)$. On the other hand we shall denote, for any open set G of $\boldsymbol{C}$, the set of analytic (i.e. single valued regular analytic) functions in G with $\mathfrak{A}(G)$. $\mathfrak{A}(G)$ forms a ring, and if $D_1 \supset D_2$, $D_i \in \mathfrak{D}(N)$, we have clearly natural homomorphisms of $\mathfrak{A}(D_1)$ in $\mathfrak{A}(D_2)$ and of $\mathfrak{A}(D_1 - N)$ in $\mathfrak{A}(D_2 - N)$. The inductive limit of rings $\mathfrak{A}(D-N)$, $D \in \mathfrak{D}(N)$, will be denoted with $\tilde{\mathcal{A}}_N$, and that of $\mathfrak{A}(D)$, $D \in \mathfrak{D}(N)$, with $\mathcal{A}_N$. $\tilde{\mathcal{A}}_N$ is then considered as an extension ring of $\mathcal{A}_N$.

The quotient $\mathcal{A}_N$-module of $\tilde{\mathcal{A}}_N$ mod. $\mathcal{A}_N$ will be denoted by $\mathcal{B}_N$, and the elements of $\mathcal{B}_N$ generally by $g(x)$. These elements will be called hyperfunctions (h. f.) on N. A h. f. $g(x)$ is given by a function $\varphi(z) \in \mathfrak{A}(D-N)$ for some $D \in \mathfrak{D}(N)$. $\varphi(z)$ is called a *defining function* of $g(x)$, and we shall write

$$(1) \qquad g(x) = \varphi(x)|^+ - \varphi(x)|^- \quad \text{or} \quad g(x) = \varphi(x+i0) - \varphi(x-i0).$$

It is easy to show that for every $g(x) \in \mathcal{B}_N$, there exists an open set $M \subset R$, such that $g(x)$ can be regarded as an element of $\mathcal{B}_M$,

1) L. Schwartz: Théorie des Distributions, I, II, Paris, Hermann (1950-1951).

2) After I had completed the manuscript of this note, I was kindly informed by Professor A. Weil through Professor Iyanaga that the same notion as that of "hyperfunction" had been already introduced by Professor G. Köthe in his paper: Die Randverteilungen analytischer Funktionen, Math. Z., **57** (1952). The content of §§ 2, 4 of the present note is also essentially contained in the paper of Professor Köthe, but the localization theorem and the extension to the case of n-dimensional manifolds are not considered in that paper.

defined by $\varphi(z)\in\mathfrak{A}(D-M)$, with $D=D^{+}+M+D^{-}$ where $D\in\mathfrak{D}(M)$, $D^{+}=\{z;\ \Im z>0,\ z\in D\}$, $D^{-}=\{z;\ \Im z<0,\ z\in D\}$. If we define then

$$(2)\qquad \varphi^{+}(z)=\begin{cases}\varphi(z) & (z\in D^{+})\\ 0 & (z\in D^{-}),\end{cases}\qquad \varphi^{-}(z)=\begin{cases}0 & (z\in D^{+})\\ \varphi(z) & (z\in D^{-}),\end{cases}$$

we have $\varphi^{+}(z),\ \varphi^{-}(z)\in\mathfrak{A}(D-M)$, and $\varphi(x)|^{+}$, $\varphi(x)|^{-}$ are just h. f. defined by $\varphi^{+}(z)$, $\varphi^{-}(z)$ of which $g(x)$ is the difference.

For example we have for $N=\{0\}$ the Dirac function $\delta(x)$ as a h. f.

$$(3)\qquad \delta(x)=-\frac{1}{2\pi i}\left(\frac{1}{x+i0}-\frac{1}{x-i0}\right).^{3)}$$

As $\mathscr{B}_N$ is an $\mathscr{A}_N$-module by definition, addition, multiplication by a complex number, and multiplication by a regular function are defined for h. f., e. g. the product of $f(x)\in\mathscr{A}_N$ and $g(x)=\varphi(x)|^{+}-\varphi(x)|^{-}\in\mathscr{B}_N$ is given by

$$f(x)g(x)=f(x)\varphi(x)|^{+}-f(x)\varphi(x)|^{-}.$$

Further, derivatives of a h. f. are defined by:

$$(4)\qquad g^{(n)}(x)=\varphi^{(n)}(x)|^{+}-\varphi^{(n)}(x)|^{-}.$$

For example, we have $\delta(x^{n})=(-1)^{n-1}(n-1)!\,\delta^{(n-1)}(x)$.

2. *Integration.* In this paragraph we consider exclusively h. f. defined on a compact set $F\subset R$. Such a h. f. is called a *particular h. f.* (p. h. f.). For a p. h. f. $g(x)=\varphi(x)|^{+}-\varphi(x)|^{-}$ on a compact F, we define the *definite integral* by

$$\int_F g(x)\,dx=-\oint_\Gamma \varphi(\zeta)\,d\zeta$$

where Γ is a rectifiable path in $D\in\mathfrak{D}(F)$ going round F in the positive sense. By Cauchy's theorem, the value of the integral does not depend on the choice of Γ and $\varphi(z)$.

For example, we have $\int_F \delta(x)\,dx=1$ whenever $F\ni 0$. As another example, we can simply deduce integral representation for Euler's $B(\alpha,\beta)$ valid for all $\alpha,\beta\neq 0,-1,-2,\cdots$:

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx=B(\alpha,\beta),$$

which is essentially Pochhammer's contour integral representation.[4)]

Now let $z\in D$. Let Γ,Γ' be rectifiable paths in D going round F in the positive sense, and suppose that z is inside Γ and outside Γ'. Let $\varphi(z)$ be a defining function of a h. f. $g(x)$. Then we have

3) More generally, for any meromorphic $f(x)$ on open M, which is $\not\equiv 0$ on each component of M, we define

$$\delta(f(x))=-\frac{1}{2\pi i}\left\{\frac{1}{f(x)}\Big|^{+}-\frac{1}{f(x)}\Big|^{-}\right\}.$$

Note that $1/f(z)$ is regular on $D-M$ for a suitable $D\in\mathfrak{D}(M)$.

4) Similarly, Hadamard's *finite part of a divergent integral* is automatically absorbed in our definition of integration.

$$\varphi(z)=\frac{1}{2\pi i}\Big(-\oint_{\Gamma'}+\oint_{\Gamma}\Big)\frac{\varphi(\zeta)}{\zeta-z}d\zeta=\Phi(z)+f(z)$$

where $f(z)=(1/2\pi i)\oint_{\Gamma}\{\varphi(\zeta)/(\zeta-z)\}d\zeta\in\mathcal{A}_F$, so that $\varphi(z)\equiv\Phi(z) \bmod \mathcal{A}_F$, and $\Phi(z)=(1/2\pi i)\int_F\{g(x)/(x-z)\}dx$ is another defining function of $g(x)$. We shall call $\Phi(z)$ the *standard defining function* of $g(x)$. As $g(x)=\Phi(x)|^+-\Phi(x)|^-$, we notice that a p. h. f. $g(x)$ defined on a compact F can be identified with a h. f. defined on any locally compact subset N of R containing F. We shall denote with $\mathcal{B}_N^*$ the $\mathcal{A}_N$-submodule of $\mathcal{B}_N$ consisting of all p. h. f. on N (i.e. h. f. which can be identified with p. h. f. defined on some compact subsets of N).

3. *Localization.* Now let M be an open subset of R, and $D\in\mathfrak{D}(M)$ such that $D=D^++M+D^-$. If $f(x)\in\mathcal{A}_M$, and $f(z)\in\mathfrak{A}(D)$, then

$$\varphi(z)=\begin{cases}f(z) & \text{for } z\in D^+\\ 0 & \text{for } z\in D^-\end{cases}$$

belongs to $\mathfrak{A}(D-M)$, and defines accordingly a h. f. on M. We can identify this h. f. with $f(x)$. In this sense we have $\mathcal{A}_M\subset\mathcal{B}_M$. A h. f. $\in\mathcal{A}_M$ is called *regular* on M.

If M' is any open subset of M and $g(x)\in\mathcal{B}_M$, we can define the *restriction* $g_{M'}(x)\in\mathcal{B}_{M'}$ of $g(x)$ in the following way. Let $g(x)=\varphi(x)|^+-\varphi(x)|^-$ with $\varphi(z)\in\mathfrak{A}(D-M)$, $D\in\mathfrak{D}(M)$. Clearly there exists a complex nbd D' of M' contained in D. The restriction of $\varphi(z)$ on D' defines a h. f. on M'. We call it $g_{M'}(x)$. Obviously this does not depend on the choice of $\varphi(z)$. We say $g(x)$ is $=0$ or is *regular on M'* according as $g_{M'}(x)=0$ or is regular on M'. Let M_0', M_1' be respectively the maximal (open) subsets of M on which $g(x)=0$ or regular. The complementary sets $M-M_0'$, $M-M_1'$ are called respectively the *carrier* of $g(x)$ and the *carrier of irregularity* of $g(x)$. The carrier is the totality of singularities of $\varphi(z)$ in D.[5]

Now let $\{M_\alpha\}$ be an open covering of M. If $g_\alpha(x)\in\mathcal{B}_{M_\alpha}$ is given on each M_α, such that any g_α and g_β have a common restriction on $M_\alpha\frown M_\beta$, we say: a h. f. in the wider sense is defined on M. Then, it can be proved that *any h. f. in the wider sense is a h. f., i.e. there exists a $g(x)\in\mathcal{B}_M$ which has $g_\alpha(x)$ as its restriction on each M_α.*[6]

On the other hand, let $g(x)$ be a h. f. on any locally compact N. Then it can be proved that there exists a defining function $\Phi(z)$ of $g(x)$ such that $\Phi(z)\in\mathfrak{A}(\boldsymbol{C}-\overline{N})$. From this fact we can easily derive the following proposition.

Let N be a locally compact subset of R and let F be a closed subset

5) E. g. $\delta(f(x))$ has the discrete carrier on M.

6) When we introduce generally h. f. on n-dim. C^ω-manifolds, a corresponding theorem plays an important rôle.

of N. A h.f. on F can be identified with a h.f. on N vanishing on $N-F$: $\mathscr{B}_F \subset \mathscr{B}_N$. We have furthermore $\mathscr{B}_{N-F} \simeq \mathscr{B}_N \bmod \mathscr{B}_F$.

Now let $\{F_n;\ n=1, 2, \cdots\}$ be a locally finite closed covering of N, $g_n(x)$ be any given h.f. on F_n. Then *by a formal sum $g(x)=\sum g_n(x)$ a h.f. in the wider sense on N is defined.* According to the above result, we have $g(x) \in \mathscr{B}_N$. Conversely, *any $g(x) \in \mathscr{B}_N$ and any locally finite closed covering $\{F_n\}$ of N being given, a decomposition $g(x)=\sum g_n(x)$ with $g_n(x) \in \mathscr{B}_{F_n}$ exists.*

4. *Duality.* Let $\{f_n(x)\}$ be a sequence of regular functions on N. We say $\{f_n(x)\}$ is *convergent in* $\mathscr{A}_n$ when these $f_n(x)$ can all be prolonged analytically into a common $D \in \mathfrak{D}(N)$, and $f_n(z) \to f(z)$ uniformly in the wider sense in D. Then $f(x) \in \mathscr{A}_N$, and $\mathscr{A}_N$ is "closed under this convergence".

We can introduce a *similar convergence concept into* $\mathscr{B}_N^*$, and using it, it is proved that (i) for $f(x) \in \mathscr{A}_N$ and $g(x) \in \mathscr{B}_N^*$, the product $f(x) \times g(x) \in \mathscr{B}_N^*$ is continuous with respect to both factors, and (ii) integration of $g(x) \in \mathscr{B}_N^*$ is a continuous operation. Thus a continuous *inner product* can be introduced by

$$(f, g)=\int_N f(x)g(x)\,dx \qquad \text{for } f(x) \in \mathscr{A}_N,\ g(x) \in \mathscr{B}_N^*$$

for which the following results are obtained.

I. *Let $T(f)$ be a continuous linear functional on $\mathscr{A}_N$, then there exists a $g(x) \in \mathscr{B}_N^*$ for which $T(f)=(f, g)$ holds for every $f(x) \in \mathscr{A}_N$.* This $g(x)$ is given by $g(x)=\Phi(x)|^+ - \Phi(x)|^-$, where $\Phi(z)=(1/2\pi i)T_x(1/(x-z))$.

II. *Let $T(g)$ be a continuous linear functional on $\mathscr{B}_N^*$, then there exists a $f(x) \in \mathscr{A}_N$ for which $T(g)=(f, g)$ holds for every $g(x) \in \mathscr{B}_N^*$.* This $f(x)$ is given by $f(x')=T_x(\delta(x-x'))$.

Thus $\mathscr{A}_N$ and $\mathscr{B}_N^*$ constitute a dual pair of linear spaces.

5. From the above results, we can see in particular that *any Schwartz's distribution T on open M with a compact carrier determines a p.h.f. on M.* In fact, $\mathscr{A}_M$ constituting a dense subspace of Schwartz's $\mathscr{E}_M$, each continuous linear functional on $\mathscr{E}_M$ can be regarded as a continuous linear functional on $\mathscr{A}_M$.

This can also be seen in the following way. It is easy to show that any C^1 function on M with a compact carrier is expressible as a h.f. By differentiating a sufficient number of times, we see by (4) that every distribution on M with a compact carrier is expressed as a h.f. By the results at the end of § 3 and the corresponding theorems for Schwartz's distributions, we infer that *any distribution with an arbitrary carrier is also a h.f.*

Now we introduce as follows the *characteristic function* $M(r)$ of any p.h.f. $g(x)$ on a compact F. Let $\Phi(z)$ be the standard defining function of $g(x)$. We put

$$M(r)=\max(\log^{+}|\Phi(z)|;\ \mathrm{dist}(z,F)=r).$$

The behavior of $M(r)$ for $r\to 0$ indicates the "complexity" of $g(x)$. Schwartz's distribution is completely characterized by $M(r)=O(\log(1/r))$. Specifically, a h. f. on a point $x=a$ is expressible as a convergent series of derivatives of $\delta(x-a)$, while a distribution corresponds to a finite series.

Thus far, we have defined h. f. as boundary values of analytic functions. Alternatively, we can define them by means of *harmonic functions*. Then we can define an *inner product* $(g_1(x), g_2(x))_x$ *of two h. f.* $g_1(x)$, $g_2(x)$ using a generalized Dirichlet integral as far as it converges, and then the *product* $g_1(x)g_2(x)=\Phi(x)|^{+}-\Phi(x)|$ where $\Phi(z)=(1/2\pi i)\ (g_1(x)/(x-z),\ g_2(x))_x$.[7]

In utilizing the theory of h. f. of two variables, we can develope a *general theory of Fourier transformations*. In particular, a h. f. $g(e^{i\theta})$ on the unit circle is expressible in the form $g(e^{i\theta})=\sum c_n e^{in\theta}$ where $\overline{\lim}_{|n|\to\infty}\sqrt[|n|]{|c_n|}\leq 1$. This g is a distribution if and only if $|c_n|=O(|n|^r)$ for some $r<\infty$.

The author wishes to end this note in expressing his thanks to Professor Iyanaga for his encouragement during the preparation of this work.

7) By this multiplication, we can explain the facts such as $(C)\cdot(C)\subset(C)$, $(L^{p_1})\cdot(L^{p_2})\subset(L^{p})$ (where $1/p_1+1/p_2=1/p<1$).