29. Total Orderings on a Semilattice[1]

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A semilattice S is called *orderable* if there exists an ordering $\leqq$ such that (1) $a<b$ implies $ac\leqq bc$ and (2) $a\leqq b$ or $b\leqq a$ for any $a, b\in S$. Such an ordering is called *permissible* on S. The main purpose of this note is to present a necessary and sufficient condition for a semilattice to be orderable.[2]

THEOREM 1. *A semilattice is orderable if and only if it does not contain any subsemilattice consisting of four elements a, b, c, d satisfying either* (i) $ab=b$, $ac=c$, $bc=d$ *or* (ii) $ab=ac=bc=d$.

COROLLARY. *Any chain, in the sense that $ab=a$ or b for all a, b, is orderable.*

An element a of a semilattice is called maximal if $ax=a$ implies $x=a$.

THEOREM 2. *Let S be an orderable semilattice. Let T be the complement of the set of all maximal elements in S. Then there exists a one to one correspondence between the set of all permissible orderings on S and the set of all subsets of T.*

COROLLARY. *Let $N(n)$ be the number of all non-isomorphic orderable semilattices consisting of n elements. Then it satisfies the following formula:*

$$N(n+1)=\sum_{\substack{0\leqq p<q\leqq n\\ p+q=n}} N(p)N(q)+\begin{cases}\frac{1}{2}N(n/2)(N(n/2)+1) & \text{if } n \text{ is even,}\\ 0 & \text{if } n \text{ is odd.}\end{cases}$$

Further, $N(n)$ is equal to the coefficient of x^n in the expansion of $f(x)$ defined by

$$f(x^2)+x(f(x))^2-2f(x)+2=0.$$

We shall close this note by listing here two examples as application.

EXAMPLE 1. Let S be a chain. Let (A, B) be a partition of S, in the sense that $S=A\smile B$, $A\frown B=\square$, the empty set. Then the ordering defined by

$$x\leqq y \quad \text{if and only if} \quad \begin{cases} x, y\in A, & xy=x\\ \text{or} \quad x\in A, & y\in B\\ \text{or} \quad x, y\in B, & xy=y,\end{cases}$$

gives a permissible ordering on S.

Conversely, any permissible ordering on S can be obtained by a

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2) This is an abstract of the paper which will appear elsewhere.

partition on S and the above formula.

The correspondence between the set of all permissible orderings and the set of all partitions is one to one except that the partitions $(S, \square)$ and $(S \backslash a, a)$ define the same ordering if S has a maximal element a.

EXAMPLE 2. Let $X = \{a, b\}$. Let S be the set of all finite sequences of elements of X. Then null sequence (=empty set) is considered as a finite sequence. Define multiplication by

$$xy = \text{the longest common cut of } x \text{ and } y.$$

Then S is a semilattice. By a numerical function defined by

$$m(x) = \sum_{i=1}^{n} (x_i)/2^i,$$

if $x = x_1 x_2 \cdots x_n$, where $x_i = a$ or b and $(a) = -1$, $(b) = 1$, $m(\square) = 0$, we can introduce an ordering

$$x < y \quad \text{if and only if} \quad m(x) < m(y).$$

Then this ordering is permissible on S.

Furthermore, any orderable finite semilattice with a permissible ordering is imbedded into this semilattice with respect to both the multiplication and the ordering.