49. On the Recurrence Theorems in Ergodic Theory

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1. For an ergodic, measure-preserving, one-to-one point transformation on a space of finite measure, M. Kac [2] made an interesting recurrence theorem which evaluates the value of the integral of a recurrence time. In this note we shall first state a recurrence theorem (Theorem 1) which enlightens the asymptotic behavior of a recurrence time. Next, on using the theorem, we shall give another proof of the Kac theorem (Theorem 2).

2. Let $(X, \mathcal{F}, \mu)$ be a measure space such that X is an abstract space, $\mathcal{F}$ a σ-field of subsets of X and μ a finite measure on $\mathcal{F}$. It is supposed that $X \in \mathcal{F}$. Let T be a measure-preserving, single-valued (not necessarily one-to-one) point transformation of X into itself, that is,

$$T^{-1}E=\{x; Tx \in E\} \in \mathcal{F} \quad \text{and} \quad \mu(T^{-1}E)=\mu(E)$$

for any $E \in \mathcal{F}$.

Before stating the definition of a recurrence time we recall a well-known

Recurrence theorem. *For every set $E \in \mathcal{F}$ we can choose a set $N \in \mathcal{F}$ of measure zero such that for each $x \in E-N$ there exists a positive integer $n(x)$ which satisfies $T^{n(x)}x \in E$* (for example, see [1], p. 10).

Let E be a set in $\mathcal{F}$. The *recurrence time* $r(x)=r(x, E)$ denotes, for each $x \in E$, the least positive integer such that $T^{r(x)}x \in E$. Then $r(x)$ is defined almost everywhere in E by virtue of the recurrence theorem stated above. Further we define $r(x)=0$ for each $x \notin E$.

Theorem 1. *For every $E \in \mathcal{F}$, the recurrence time $r(x)$ is an integrable function and*

$$(1) \qquad \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} r(T^j x) \begin{cases} =1 & \text{for almost all } x \in E, \\ \leqq 1 & \text{for almost all } x \notin E. \end{cases}$$

Theorem 2. *For every $E \in \mathcal{F}$,*

$$(2) \qquad \mu(E) \leqq \int_E r(x)\mu(dx) \leqq \mu(X).$$

Moreover, T is ergodic if and only if

$$(3) \qquad \int_E r(x)\mu(dx)=\mu(X)$$

for every $E \in \mathcal{F}$ of positive measure.

3. Before proving the theorems we prepare a lemma and a remark.

Lemma. *If, for a non-negative measurable function $f(x)$, there exists an integrable function $h(x)$ such that*

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} f(T^j x)\leqq h(x) \quad \textit{for almost all } x,$$

then $f(x)$ is an integrable function.

Proof. Let $f_k(x)=\min(f(x), k)$ $(k=1, 2,\cdots)$. Then each $f_k(x)$ is integrable, since μ is a finite measure. By the individual ergodic theorem there exists an integrable function $\tilde{f}_k(x)$ such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} f_k(T^j x)=\tilde{f}_k(x) \quad \text{for almost all } x$$

and

$$\int_X \tilde{f}_k(x)\mu(dx)=\int_X f_k(x)\mu(dx).$$

Since $\tilde{f}_k(x)\leqq h(x)$ and $\{f_k(x)\}$ is a monotone increasing sequence and converges to $f(x)$, it follows, by the convergence theorem, that

$$\begin{aligned}\int_X f(x)\mu(dx)&=\lim_{k\to\infty}\int_X f_k(x)\mu(dx)\\ &=\lim_{k\to\infty}\int_X \tilde{f}_k(x)\mu(dx)\leqq\int_X h(x)\mu(dx)<\infty,\end{aligned}$$

that is, $f(x)$ is an integrable function.

Remark for the recurrence time. Theorems 1 and 2 are not influenced by the modification of values of $r(x)$ for x in a set of measure zero. For a given set E, let N denote the exceptional set of measure zero in the recurrence theorem. If we set $\tilde{N}=\bigcup_{j=0}^{\infty} T^{-j}N$, then $\mu(\tilde{N})=0$. We have nothing particular to say about the values of $r(x)$ in a set $E-\tilde{N}$. However, we define newly $r(x)=0$ for x in a set $E\frown\tilde{N}$. If $x\notin\tilde{N}$, there exists no positive integer n such that $T^n x\in\tilde{N}$. Therefore, if we define, for each $x\in E-\tilde{N}$, $r_1(x)=r(x)$, $r_2(x)=r_1(x)+r(T^{r_1(x)}x),\cdots$, $r_n(x)=r_{n-1}(x)+r(T^{r_{n-1}(x)}x),\cdots$, then we have

$$r_1(x)<r_2(x)<\cdots<r_n(x)<\cdots,$$

since $T^{r_n(x)}x\in E-\tilde{N}$ $(n=1, 2,\cdots)$.

4. **Proof of Theorems 1 and 2.** Taking any fixed $E\in\mathfrak{F}$, we consider $r(x)=r(x, E)$ under the above remark. Then

$$\{x;\, r(x)=0\}=(X-E)\smile\tilde{N}\in\mathfrak{F},$$

$$\{x;\, r(x)=k\}=(E-\tilde{N})\frown\bigcap_{j=1}^{k-1}T^{-j}((X-E)\smile\tilde{N})\frown T^{-k}(E-\tilde{N})\in\mathfrak{F} \quad (k=1, 2,\cdots).$$

Hence $r(x)$ is a measurable function.

If $x \in E-\tilde{N}$, then

$$r(x)+r(Tx)+\cdots+r(T^{r_1(x)-1}x)=r_1(x),$$
$$\cdots\cdots\cdots\cdots$$
$$r(x)+r(Tx)+\cdots+r(T^{r_n(x)-1}x)=r_n(x),$$
$$\cdots\cdots\cdots\cdots,$$

so that

$$\lim_{n\to\infty}\frac{1}{r_n(x)}\sum_{j=0}^{r_n(x)-1} r(T^j x)=1.$$

Hence, if $x \in E-\tilde{N}$,

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} r(T^j x)\geqq \lim_{n\to\infty}\frac{1}{r_n(x)}\sum_{j=0}^{r_n(x)-1} r(T^j x)=1,$$

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} r(T^j x)\leqq \lim_{n\to\infty}\frac{1}{r_n(x)}\sum_{j=0}^{r_n(x)-1} r(T^j x)=1.$$

Next, assume $x \notin E-\tilde{N}$. If there exists no positive integer n such that $T^n x \in E-\tilde{N}$,

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} r(T^j x)=0.$$

If there exists a positive integer n such that $T^n x \in E-\tilde{N}$, then by $p(x)$ we denote the least positive integer such that $T^{p(x)}x \in E-\tilde{N}$. Set $y=T^{p(x)}x$. Then

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} r(T^j x)\leqq \liminf_{n\to\infty}\frac{1}{p(x)+r_n(y)}\sum_{j=0}^{p(x)+r_n(y)-1} r(T^j x)$$
$$=\liminf_{n\to\infty}\frac{1}{r_n(y)}\cdot\frac{r_n(y)}{p(x)+r_n(y)}\left\{\sum_{j=0}^{p(x)-1} r(T^j x)+\sum_{j=0}^{r_n(y)-1}(T^j y)\right\}$$
$$=\liminf_{n\to\infty}\frac{1}{r_n(y)}\sum_{j=0}^{r_n(y)-1} r(T^j y)=1.$$

Consequently, we have

$$(4)\qquad \limsup_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} r(T^j x)\geqq 1 \quad \text{for almost all } x\in E,$$

$$(5)\qquad \liminf_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} r(T^j x)\leqq 1 \quad \text{for almost all } x\in X.$$

By virtue of the lemma and (5), $r(x)$ is an integrable function. Hence, by the individual ergodic theorem, there exists an integrable function $\tilde{r}(x)$ such that

$$(6)\qquad \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} r(T^j x)=\tilde{r}(x) \quad \text{for almost all } x\in X,$$

$$(7)\qquad \int_X \tilde{r}(x)\mu(dx)=\int_X r(x)\mu(dx).$$

By (4), (5) and (6) we have

$$(8)\qquad \begin{aligned}\tilde{r}(x)&=1 \quad \text{for almost all } x\in E,\\ &\leqq 1 \quad \text{for almost all } x\in X-E,\end{aligned}$$

which is just (1). Thus the proof of Theorem 1 is terminated.

Further, by (7) and (8) we have

$$\mu(E)=\int_E 1\mu(dx)\leqq\int_X \tilde{r}(x)\mu(dx)=\int_X r(x)\mu(dx)$$
$$=\int_E r(x)\mu(dx)=\int_X r(x)\mu(dx)=\int_X \tilde{r}(x)\mu(dx)$$
$$\leqq\int_X 1\mu(dx)=\mu(X),$$

which gives (2).

Next, assume that T is ergodic. Take any set $E\in\mathcal{F}$ of positive measure. Since T is ergodic, $r(x, E)$ must be constant almost everywhere. Hence, by (8), $\tilde{r}(x)=1$ almost everywhere, so that we obtain (3).

Conversely, assume that T is not ergodic. Then there exists an invariant set E such that $\mu(E)>0$ and $\mu(X-E)>0$. Since $T^{-n}E=E$ and $T^{-n}(X-E)=X-E$, we have that $T^n x\in E$ $(n=1,2,\cdots)$ for all $x\in E$ and $T^n x\in X-E$ $(n=1,2,\cdots)$ for all $x\in X-E$, so that

$$\begin{aligned} r(x)&=1 \quad \text{for all } x\in E \\ &=0 \quad \text{for all } x\in X-E. \end{aligned}$$

Since both E and $X-E$ are of positive measures, (3) does not hold. Thus the proof of Theorem 2 is terminated.

5. For any fixed set $E\in\mathcal{F}$, we set

$$\tilde{E}=\bigcup_{j=0}^{\infty} T^{-j}E.$$

Then, for $x\in\tilde{E}$, there exists a positive integer n such that $T^n x\in E$ and, for $x\notin\tilde{E}$, there exists no positive integer n such that $T^n x\in E$. Hence, on modifying Theorems 1 and 2, we obtain

Theorem 3. *For any set $E\in\mathcal{F}$, the recurrence time $r(x)$ is an integrable function and*

$$\begin{aligned} \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} r(T^j x)&=1 \quad \textit{for almost all } x\in\tilde{E}, \\ &=0 \quad \textit{for almost all } x\notin\tilde{E}, \end{aligned}$$

and

$$\int_E r(x)\mu(dx)=\mu(\tilde{E}).$$

References

[1] P. R. Halmos: Lectures on ergodic theory, Tokyo (1956).
[2] M. Kac: On the notion of recurrence in discrete stochastic processes, Bull. Amer. Math. Soc., **63**, 1002–1010 (1947).