59. A Remark on a Subdomain of a Riemann Surface of the Class O_{HD}

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In this paper we shall use the compact Hausdorff space due to H. L. Royden. Let W be an open Riemann surface and BD the class of piecewise smooth functions defined on W and bounded on it having a finite Dirichlet integral $D[f]$. To make use of the theory of normed ring, we introduce in BD a new norm given by

$$||f||=\sup|f|+\sqrt{D[f]}.$$

We denote by BD^* the completion of BD by means of this norm. Then BD^* is a normed ring A. The set of the maximal ideals constructs a compact Hausdorff space W^* by means of a certain topology [1], and then W is embedded in W^* as open and dense subset. We denote by K the class of BD with compact carrier, and denote by $\overline{K}$ the class of functions which are limits in BD of sequences from K. Next, we denote by $\Gamma(W)$ or $\Delta(W)$ the set of the maximal ideals which contain K or $\overline{K}$ respectively, and we call $\Gamma(W)$ the ideal boundary of F, $\Delta(F)$ the harmonic boundary of F [3]. We notice that all functions of BD are extended continuously on W^* [1], and all functions of class HBD attain its maximum and minimum on Δ, consequently it is determined uniquely by the distribution of the values on $\Delta(F)$ [5]. We shall prove the following

Theorem. If $F\in O_{HD}-O_G$, any non-compact domain G of F belongs to the class SO_{AD}.

Proof. By $G\in SO_{AD}$, we designate that all functions which are continuous on $\overline{G}$ and belong to class AD on G must reduce to a constant, provided that its real part vanishes on the relative boundary of G.

First of all, we notice that $\Delta(F)$ consists of only one point. Because, by our assumption there are no non-constant HBD on F, consequently $\overline{K}$ is the maximal ideal of A. In the following we intend to lead to contradiction by denying our proposition.

Suppose that $G\overline{\in}SO_{AD}$, then there exists an analytic function $f=u+iv$ such that it is analytic on G, continuous on $\overline{G}$ (closure of G) and u vanishes on the relative boundary of G.

Since the number of the branch points is countable, there is a level curve of u on which there are no branch points. Let L'_λ be the level curve of $u(p)$ with height λ on which there are no branch points, and let G_λ be the subdomain of G such that the relative boundary of G_λ consists of some components of L'_λ. Then u is of the class HD, because f is of the class AD. According to A. Mori's theorem, u is of the class HBD, and it attains either maximum or minimum on the relative boundary of G_λ [4]. Without loss of generality, we can assume that $u>\lambda$ in G_λ. Let L_λ be the component of the relative boundary of G_λ and let $U(p_0)$ be a neighbourhood of $p_0(\in L_\lambda)$ which is an image of parameter disc. We now consider a level curve $L(v)$ of v passing p_0. Let $L^0(v)$ be the connected part of $L(v)$ which is in $G_\lambda \cap U(p_0)$ and contains p_0.

Next, we take on $L^0(v)$ such a point q_0 that satisfies the following condition: there are no branch points on the arc $\widehat{p_0 q_0}$, which is the connected part of $L^0(v)$. This is possible, because the number of the branch points is countable. Next let $L(p)$ be the open arc of the level curve of u passing p which is any inner point of $\widehat{p_0 q_0}$, and suppose that there are no branch points on $L(p)$ except possibly its both ends. Let D be the point-set consisting of all $L(p)$. Then we can prove that D is a domain and u attains constants respectively on each components of the relative boundary of D [6].

The analytic function $f=u+iv$ is univalent in D. For v is monotone on each $L(p)$. Thus we can consider the conformal image of D in the complex ζ-plane by means of $if=-v+iu$ $(=\xi+i\eta)$. Let D_ζ be the conformal image of D. Each component of the relative boundary of D consists of a half-straight line or a segment parallel to η-axis. Let $\tilde{L}_\lambda$ and $\tilde{L}_\mu$ be respectively the image of $L(p_0)$ and $L(q_0)$.

We put a cut along each relative boundary component of D except $\tilde{L}_\lambda$ and $\tilde{L}_\mu$, and we construct the double $\hat{D}_\zeta$ by combining the upper or lower side of the cut respectively with the upper or lower side of the other domain indirectly conformal to D_ζ. This double is of hyperbolic type and does not belong to O_{HD}. For symmetric extension of $-v=R[if]$ is harmonic on $\hat{D}_\zeta$ and its Dirichlet integral with respect to $\hat{D}_\zeta$ is finite.

We denote by $\hat{D}^*_\zeta$ the compactification of $\hat{D}_\zeta$ by means of Royden's method. We notice that the ideal boundary points of $\hat{D}_\zeta$ corresponding to the end-point of the cuts are not the harmonic boundary points. For we can know easily that the end-point of the cut is the removal singular point with respect to HBD on $\hat{D}_\zeta$, consequently any element of HBD does not attain neither the maximum nor minimum at this point. Now each end-point of the cut is separated from the other end-points by a certain closed curve, consequently the ideal boundary points of $\hat{D}_\zeta$ corresponding to each end-point of the cut are respectively separated from the other.

Let ζ_0 be an end-point of a cut, and let $\mathfrak{F}(\zeta_0)$ be the set of the ideal boundary points corresponding to ζ_0. If $\mathfrak{F}(\zeta_0)$ contains the harmonic boundary points, we can construct the function f ($\in HBD(\hat{D}_\zeta)$) such as

$$f=0 \qquad \text{on } \Delta(\hat{D}_\zeta)-\mathfrak{F}(\zeta_0)$$
$$\neq 0 \qquad \text{on } \mathfrak{F}(\zeta_0),$$

here $\Delta(\hat{D}_\zeta)$ is the harmonic boundary of $\hat{D}_\zeta$.

We decompose f^2, i.e. [2]

$$f^2=u+\varphi, \quad u\in HBD(\hat{D}_\zeta), \quad \varphi\in \overline{K}(\hat{D}_\zeta),$$

then $u=f^2$ on $\Delta(\hat{D}_\zeta)$, since $\varphi\equiv 0$ on $\Delta(\hat{D}_\zeta)$. This is absurd. In fact according to our notice, the function of class $HBD(\hat{D}_\zeta)$ does not take neither maximum nor minimum on $\mathfrak{F}(\zeta_0)$.

Next let φ be an element of $\overline{K}(F)$ with respect to F, and let $\hat{\varphi}$ be the symmetric extension of φ to $\hat{D}_\zeta$. Then $\hat{\varphi}$ belongs to $\overline{K}(\hat{D}_\zeta)$ with respect to $\hat{D}_\zeta$. We shall give the reason in the following.

Let $\{\varphi_n\}_1^\infty$ be a sequence such that each φ_n belongs to $K(F)$, i.e. of which carrier compact, and $\varphi_n\to\varphi$ (in BD). We construct $\hat{\varphi}_n$ which is the symmetric extension of φ_n to $\hat{D}_\zeta$. Then $\hat{\varphi}_n$ belongs to $\overline{K}(\hat{D}_\zeta)$ and $\hat{\varphi}_n\to\hat{\varphi}$ (in BD). The latter is evident, therefore we shall verify the former. According to the Royden's decomposition

$$\hat{\varphi}_n=S+\chi, \quad S\in HBD(\hat{D}_\zeta), \quad \chi\in\overline{K}(\hat{D}_\zeta),$$

and $\hat{\varphi}_n$ vanishes on the harmonic boundary of $\hat{D}_\zeta$, because there are no harmonic boundary points corresponding to the end-points of the cuts. Therefore $S=0$ on the harmonic boundary of $\hat{D}_\zeta$, hence $S\equiv 0$ on $\hat{D}_\zeta$ [5]. Thus we know that $\hat{\varphi}_n$ belongs to $\overline{K}(\hat{D}_\zeta)$. From this we see that $\hat{\varphi}$ belongs to $\overline{K}(\hat{D}_\zeta)$, here φ is any element of the class $\overline{K}(F)$.

Next we shall show that $\overline{D}\cap\Delta(F)$ is not empty set, here $\overline{D}$ is the closure of D with respect to F^* (compactification of F), and $\Delta(F)$ is the harmonic boundary. Suppose that $\overline{D}\cap\Delta(F)$ is empty set. Then at each point M of $\overline{D}-D$, there exists an element φ of $\overline{K}(F)$ such that $\varphi(M)\neq 0$. Let $U(M)$ be a neighbourhood of M such that

$$U(M)=\{\tilde{M};\varphi^2(M)>\varphi^2(\tilde{M})-\varepsilon>0:\tilde{M}\in F^*\}.$$

Applying the finite covering property, we can construct the function $\psi\in BD(F)$ such that ψ is non-negative and does not vanish on $\overline{D}-D$. Now let $g(p;q)$ be the Green function of F, here $q\bar{\in} D$. Then $\chi\equiv g+\psi$ is of the class $\overline{K}(F)$, consequently $\hat{\chi}\in\overline{K}(\hat{D}_\zeta)$, and from this we see that $\inf\hat{\chi}=0$ on $\hat{D}_\zeta$. Considering that $\hat{\chi}$ is the symmetric extension of χ, we see that $\inf \chi=0$ on D_{ζ}. Thus we conclude that $\inf \chi=0$ on $\overline{D}$, as χ is continuous on F^* [1]. While $\inf \chi>0$ on $\overline{D}$, as $\overline{D}$ is compact. This is absurd. Thus we have shown that $\overline{D}\cap\Delta(F)$ is not empty.

Next, we define the function $\tilde{u}$ on F, such that

$$\begin{aligned} \tilde{u}&=u \qquad \text{on } G_\lambda \\ &=\lambda \qquad \text{on } F-G_\lambda. \end{aligned}$$

As this function belongs to $BD(F)$, consequently

$$\tilde{u}=c+\varphi,$$

where c is a constant and $\varphi\in\overline{K}(F)$.

Since $u>\lambda$ on G and takes a constant λ on the relative boundary of G_λ, u is subharmonic on F. By considering the process of the orthogonal decomposition, we see that $\tilde{u}<c$ on F, hence $\varphi<0$ on F. Thus we know that

$$\sup_F \tilde{u}=\sup_{G_\lambda} u=c,$$

and the value c is attained by u at the harmonic boundary point of F. While $\overline{D}$ contains the harmonic boundary point of F and $\lambda\leq u\leq\mu$ on D, and so also on $\overline{D}$ because $u\in BD(F)$. This is absurd. Thus we have verified the theorem.

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