54. *Second Order Linear Ordinary Differential Equations Containing a Large Parameter*

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1. **Introduction.** In a number of papers, R. E. Langer has presented a general method for constructing the asymptotic solutions of linear ordinary differential equations containing a large parameter. For the case of the second order, his method has been successfully applied to the turning points of order either one or two, while, because of the difficulty in constructing the so-called *related differential equations* as well as the complexity in the analysis, the existing theories are still incomplete for the turning points of higher order.[1]

As a matter of fact, these difficulty and complexity are rather intrinsic. Still, a much simpler treatment of the problem is possible for a differential equation of the form

(1.1) $$d^2y/dx^2+(\lambda^2\phi(x)+R(x,\lambda))y=0,$$

with which we shall be concerned in this paper.

2. **Basic assumptions.** We shall start by giving our basic assumptions precisely. We assume that the variable x and the parameter λ are complex; $R(x,\lambda)$ is supposed to be a function holomorphic in x and λ, having an asymptotic expansion

(2.1) $$R(x,\lambda)\simeq\sum_{k=0}^{\infty}R_k(x)\lambda^{-k}$$ [2]

in the region

(2.2) $$|x|<\delta_0,\quad |\lambda|>\rho_0,\quad |\arg\lambda|<\alpha_0,\quad (\delta_0,\rho_0,\alpha_0>0),$$

whereas the functions $\phi(x)$ and $R_k(x)$ $(k=0,1,2,\cdots)$ to be holomorphic in x $(|x|<\delta_0)$.

The case $\phi(x)\equiv 0$ will be excluded, because this is trivial, as is obvious. Let m be the order of zero of the function $\phi(x)$ at $x=0$. If $m>0$, the point $x=0$ is a turning point of order m by definition. On the other hand, $m=0$ implies $\phi(0)\neq 0$.

By way of normalization such as the substitution of the form

$$\xi=\Phi(x)=\left\{\frac{m+2}{2}\int_0^x\phi(x)^{\frac{1}{2}}\,dx\right\}^{\frac{2}{m+2}},\quad y=u\exp\left\{-\frac{1}{2}\int\frac{\Phi''}{\Phi'^2}d\xi\right\},$$

we may assume, without loss of generality, that $\phi(x)\equiv x^m$. Accordingly, the equation (1.1) can be given the form

(2.3) $$d^2y/dx^2+(\lambda^2x^m+R(x,\lambda))y=0,$$

where m is a non-negative integer.

1) For these and terminologies, see Langer [1–4], R. W. McKelvey [1] or W. Wasow [1].

2) We denote by $\simeq$ an asymptotic relation, while $\approx$ means an equality in the formal sense.

3. Algorithm. In the asymptotic theory of the equation (2.3), it is the matter of primary importance to establish a suitable algorithm for computing the formal solutions. The guiding principle in the algorithm of Langer's for the turning points of order one may apply to other cases. Such a generalization has been given by McKelvey for the turning points of order two.[3] Upon making use of the same principle, *we can reduce the equation* (2.3) *to an equation of the type*

$$(3.1)\qquad d^2u/dx^2+(\lambda^2x^m+a_0(\lambda)+a_1(\lambda)x+\cdots+a_{m-2}(\lambda)x^{m-2})u=0$$

by the formal substitution of the form

$$(3.2)\qquad y\approx A(x,\lambda)u+B(x,\lambda)\lambda^{-1}\,du/dx,$$

where A *and* B *are formal power series in* λ^{-1} *with coefficients holomorphic in* x, *and the functions* $a_h(\lambda)$ $(h=0,1,\cdots,m-2)$ *are holomorphic in* λ, *having asymptotic expansions*

$$(3.3)\qquad a_h(\lambda)\simeq\sum_{k=0}^{\infty}S_{hk}\lambda^{-k}$$

for $|\lambda|>\rho_0$ *and* $|\arg\lambda|<\alpha_0$, *the coefficients* S_{hk} *being constants.*

To prove this statement, we note first that, if we set

$$(3.4)\qquad S(x,\lambda)=a_0(\lambda)+a_1(\lambda)x+\cdots+a_{m-2}(\lambda)x^{m-2}\simeq\sum_{k=0}^{\infty}S_k(x)\lambda^{-k},$$

where

$$S_k(x)=\sum_{h=0}^{m-2}S_{hk}x^h,$$

the equation (3.1) takes the form

$$(3.5)\qquad d^2u/dx^2+(\lambda^2x^m+S(x,\lambda))u=0.$$

Let

$$(3.6)\qquad A(x,\lambda)\approx\sum_{k=0}^{\infty}A_k(x)\lambda^{-k},\quad B(x,\lambda)\approx\sum_{k=0}^{\infty}B_k(x)\lambda^{-k}.$$

Then, differentiating (3.2) with respect to x, eliminating d^2u/dx^2 by means of (3.5), inserting the derived formulas into the equation (2.3) and equating to zero the coefficients of u and $\lambda^{-1}du/dx$, we obtain the following sequence of equations

$$(3.7)\qquad dA_k/dx=H_k+B_0S_{k-1},\quad 2x^m dB_k/dx+mx^{m-1}B_k=K_k-A_0S_{k-1},$$

where H_k and K_k are polynomials in A_p, B_p, R_p, S_q $(p<k, q<k-1)$ and their derivatives with respect to x. In particular $H_0=K_0\equiv 0$.

It would be not difficult to see how to determine the functions A_k, B_k and S_{k-1} by means of (3.7). For example, $A_0=1$, $B_0=0$ and $K_1-S_0=O(x^{m-1})$. Generally speaking, the polynomial S_{k-1} should be determined so as to satisfy the condition $K_k-S_{k-1}=O(x^{m-1})$. The equations (3.7) will then admit of solutions holomorphics in x. We can thereby determine the functions A_k and B_k.

So far the series $S(x,\lambda)$ has been purely formal. But the existence of the functions $a_h(\lambda)$ having the asymptotic expansions (3.3) in the prescribed region is well known. The statement in italics is thereby proved.

4. Main theorem. Now we can state our main theorem.

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3) See Langer [4] and McKelvey [1].

Theorem. *For any small positive constant* ε, *there exist two holomorphic functions* $A(x,\lambda)$ *and* $B(x,\lambda)$, *being represented asymptotically by the formal series* (3.6) *in a region*

$$|x|<\delta,\quad |\lambda|>\rho,\quad |\arg\lambda|<\alpha,\quad |\arg\xi|<\pi-\varepsilon, \tag{4.1}$$

such that the substitution of the form

$$y=A(x,\lambda)u+B(x,\lambda)\lambda^{-1}du/dx \tag{4.2}$$

transforms the equation (2.3) *into* (3.1), *where*

$$\xi=\frac{2}{m+2}\lambda x^{\frac{m+2}{2}}, \tag{4.3}$$

δ, ρ *and* α *being suitable positive constants.*

By rotating x and λ around $x=0$ and $\lambda=\infty$ in a suitable way, we may obtain analogous results outside the region (4.1).

To prove this theorem, we shall make use of the method which Prof. M. Hukuhara has presented in one of his papers.[4]

5. The equation. $d^2v/dx^2+\lambda^2x^mv=0$. At the outset of the proof, we shall be concerned here with the equation of the form

$$d^2v/dx^2+\lambda^2x^mv=0, \tag{5.1}$$

whose fundamental solutions can be given in the form.

$$v_j=\xi^\nu H_\nu^{(j)}(\xi)\quad\text{with}\quad \nu=1/(m+2),\quad (j=1,2), \tag{5.2}$$

where

$$H_\nu^{(j)}(\xi)=\frac{(-1)^{j+1}i}{\sin\nu\pi}\{e^{(-1)^j\nu\pi i}J_\nu(\xi)-J_{-\nu}(\xi)\}, \tag{5.3}$$

$J_\nu(\xi)$ and $J_{-\nu}(\xi)$ being the Bessel functions of the order ν.

Let

$$\Phi(\xi,\lambda)=\begin{bmatrix} v_1, & v_2 \\ \lambda^{-1}dv_1/dx, & \lambda^{-1}dv_2/dx\end{bmatrix},\quad C_0(x)=\begin{bmatrix} 0, & 1 \\ -x^m, & 0\end{bmatrix}$$

and

$$\Psi(\xi,\lambda)=\Phi\exp(-i\xi\Lambda),\quad\text{where}\quad \Lambda=\begin{bmatrix} 1, & 0 \\ 0, & -1\end{bmatrix}. \tag{5.4}$$

It is evident that

$$d\Phi(\xi,\lambda)/dx=\lambda C_0(x)\Phi(\xi,\lambda) \tag{5.5}$$

and

$$\det\Phi(\xi,\lambda)=\det\Psi(\xi,\lambda)=L_m\lambda^{2\nu-1}, \tag{5.6}$$

L_m being a certain constant distinct from zero. On the other hand, by use of the well-known asymptotic expansions of the Hankel functions at $\xi=\infty$[5] together with the formulas

$$dv_j/dx=\lambda x^{\frac{m}{2}}\xi^\nu H_{\nu-1}^{(j)}(\xi),$$

we find an inequality

$$\|\Psi(\xi,\lambda)\|\leqq M\ ^{6)} \tag{5.7}$$

in the region $|x|<\delta_0$, $|\arg\xi|<\pi-\varepsilon$, M being a suitable positive constant. By virtue of (5.6) and (5.7), if we put

$$\Psi^{-1}=\frac{\lambda^{1-2\nu}}{L_m}\Gamma(\xi,\lambda),$$

4) See Hukuhara [1].
5) See T. Inui [1, pp. 372–373].
6) Let $X=(x_{jk})$, $(j,k=1,\cdots,n)$. Then, $\|X\|=\max_{j,k}|x_{jk}|$.

we also find another inequality

(5.8) $$\| \Gamma(\xi, \lambda) \| \leqq M$$

in the same region.

6. **Fundamental equations.** A differentiation of (3.2), followed by the elimination of d^2u/dx^2 through the use of (3.5), yields

(6.1) $$\lambda^{-1}dy/dx \approx (\lambda^{-1}dA/dx-(x^m+\lambda^{-1}S)B)u+(A+\lambda^{-1}dB/dx)\lambda^{-1}du/dx.$$

We can then regard the formal relations (3.2) and (6.1) as a formal transformation of the matrix equation

$$dY/dx=[\lambda C_0(x)+\lambda^{-1}C(x, \lambda)]\,Y$$

into the equation

$$dU/dx=[\lambda C_0(x)+\lambda^{-1}D(x, \lambda)]\,U,$$

where

$$C(x, \lambda)=\begin{bmatrix} 0, & 0 \\ -R, & 0 \end{bmatrix}, \qquad D(x, \lambda)=\begin{bmatrix} 0, & 0 \\ -S, & 0 \end{bmatrix},$$

Y and U being indeterminate square matrices of order two. Let

(6.2) $$P(x, \lambda) \approx \sum_{k=0}^{\infty} P_k(x)\lambda^{-k}$$

be the matrix of this formal transformation. Since $A_0=1$ and $B_0=0$, the matrix $P_0(x)$ is the unit matrix of order two.

(5.4), (5.5) and the relation

$$dP/dx \approx (\lambda C_0+\lambda^{-1}C)P-P(\lambda C_0+\lambda^{-1}D)$$

imply that the matrix

(6.3) $$Q \approx \Psi(\xi, \lambda)^{-1}\,P(x, \lambda)\,\Psi(\xi, \lambda)$$

is a formal solution of the equation

(6.4) $$dQ/dx=i\lambda x^{\frac{m}{2}}(\Lambda Q-Q\Lambda)+\lambda^{-1}(\tilde{C}Q-Q\tilde{D}),$$

where

$$\tilde{C}=\Psi^{-1}C\Psi, \quad \tilde{D}=\Psi^{-1}D\Psi.$$

The inequalities (5.7) and (5.8) yield the inequalities

(6.5) $$\| \tilde{C}(x, \lambda) \| \leqq M_0 |\lambda|^{1-2\nu}, \quad \| \tilde{D}(x, \lambda) \| \leqq M_0 |\lambda|^{1-2\nu}$$

in the region

(6.6) $$|x|<\delta_0, \quad |\lambda|>\rho_0, \quad |\arg \lambda|<\alpha_0, \quad |\arg \xi|<\pi-\varepsilon,$$

M_0 being a suitable positive constant.

Let q_1, q_2, q_3 and q_4 be four elements of the matrix Q, arranged in a suitable order. The equation (6.4) then can be written in the form

(6.7) $$dq_j/dx=\lambda g_j(x)q_j+\lambda^{-1}\sum_{k=1}^{4} f_{jk}(x, \lambda)q_k \quad (j=1,2,3,4),$$

where

(6.8) $$g_j(x)=\begin{cases} 2ix^{\frac{m}{2}} & (j=1), \\ -2ix^{\frac{m}{2}} & (j=2), \\ 0 & (j=3,4). \end{cases}$$

The coefficients $f_{jk}(x, \lambda)$ satisfy the inequalities

(6.9) $$|f_{jk}(x, \lambda)| \leqq 2M_0 |\lambda|^{1-2\nu}$$

for (6.6).

7. Paths of integration. Let $\eta=x^{\frac{m+2}{2}}$. We shall denote by $\mathfrak{D}(\gamma)$ the interior of a square in the η-plane of which vertices are the points $\eta=\pm\gamma$ and $\pm i\gamma$, where γ is a positive constant.

Corresponding to $\mathfrak{D}(\gamma)$, there exists a square in the ξ-plane for each value of λ such that $|\lambda|>\rho$, $|\arg\lambda|<\alpha$. If α is a positive constant less than $\pi/4$, the points $\xi^*=\frac{2}{m+2}i\lambda\gamma$ and $\xi_*=-\frac{2}{m+2}i\lambda\gamma$ are respectively the uppermost and the lowermost vertices of such a square. Furthermore, for suitable positive constants γ, ρ, α and ε', the region

$$\eta\in\mathfrak{D}(\gamma),\quad |\gamma|>\rho,\quad |\arg\lambda|<\alpha,\quad |\arg\eta|<\pi-\varepsilon', \tag{7.1}$$

is contained in the region (6.6).

Let x^* and x_* be the points in the x-plane corresponding respectively to ξ^* and ξ_*. We shall then denote by Γ_{1x} a path from x_* to a point x in the region

$$\eta\in\mathfrak{D}(\gamma),\quad |\arg\eta|<\pi-\varepsilon', \tag{7.2}$$

while a path from x^* to x will be denoted by Γ_{2x}. Finally, the paths Γ_{3x} and Γ_{4x} will be taken from 0 to x. Each path must be confined to the region (7.2) except its starting point.

8. Existence of solutions. Hereafter, we shall put

$$x_j=\begin{cases}x_*\\ x^*\\ 0\end{cases}\quad\text{and}\quad \xi_j=\begin{cases}\xi_* & (j=1),\\ \xi^*, & (j=2),\\ 0 & (j=3,4).\end{cases}$$

As is well known, there exist matrices $\Pi_j(\lambda)$ holomorphic in λ and having asymptotic expansions.

$$\Pi_j(\lambda)\simeq\sum_{k=0}^{\infty}P_k(x_j)\lambda^{-k} \tag{8.1}$$

for $|\lambda|>\rho_0$, $|\arg\lambda|<\alpha_0$. For each j, let $p_j(\lambda)$ be the element of the matrix $\lambda^{2\nu-1}\Psi(\xi_j,\lambda)^{-1}\Pi_j(\lambda)\Psi(\xi_j,\lambda)$, corresponding to the element q_j of the matrix Q.

Now the matter of primary concern is to prove the existence of a bounded solution of the equations (6.7) which satisfies the conditions

$$q_j(x_j)=p_j(\lambda)\quad (j=1,2,3,4). \tag{8.2}$$

To this end, we shall make use of the well-known *fixed point theorem.*[7]

Note first that any solution of (6.7) can be represented by a set of four functions q_j. We shall then denote by $\mathfrak{F}$ a family of sets of four functions $\phi_j(x,\lambda)$ $(j=1,2,3,4)$ which are holomorphic in x and λ, and satisfy the inequalities

$$\|\phi_j(x,\lambda)\|\leqq K\quad (j=1,2,3,4) \tag{8.3}$$

in the region (7.1), where K is a positive constant independent of each member of the family $\mathfrak{F}$. With the topology of uniform convergence, *$\mathfrak{F}$ is convex, closed and compact.*

Corresponding to each member $(\phi_1,\phi_2,\phi_3,\phi_4)$ of $\mathfrak{F}$, we can define another set of functions $(\varphi_1,\varphi_2,\varphi_3,\varphi_4)$ by means of the formulas

7) See Hukuhara [2].

$$(8.4)\qquad \varphi_j(x,\lambda)=e^{\lambda G_j(x)}\left\{p_j(\lambda)+\lambda^{-1}\sum_{k=1}^{4}\int_{\Gamma_{jx}} f_{jk}(t,\lambda)\,\phi_k(t,\lambda)\,e^{-\lambda G_j(t)}\,dt\right\},$$

where

$$(8.5)\qquad G_j(x)=\int_{\Gamma_{jx}} g_j(t)\,dt.$$

By virtue of (6.9), (8.3) and the definition of $p_j(\lambda)$, upon choosing suitable paths Γ_{jx}, it would be readily seen that, if ρ and K are sufficiently large, the sets (φ_j) also belong to the family $\mathfrak{F}$. Since the mapping defined by (8.4) is continuous with respect to (ϕ_j), there exists at least a member of $\mathfrak{F}$ such that $\varphi_j\equiv\phi_j$, as is derived from the fixed point theorem. Thus, we obtain a solution of (6.7) satisfying the conditions (8.2) and the inequalities (8.3) in the region (7.1).

9. Asymptotic properties. Let $Q(x,\lambda)$ be the square matrix of order two corresponding to the solution (ϕ_j) of the equations (6.7) obtained in §8. The matrix $Q(x,\lambda)$ actually satisfies the equation (6.4), whereas the matrix (6.3) is a formal solution of the same equation. Therefore, if we put $W(x,\lambda)=Q-\lambda^{2\nu-1}\Psi^{-1}P^{(N)}(x,\lambda)\Psi$, where $P^{(N)}=\sum_{k=0}^{N-1}P_k(x)\lambda^{-k}$ $(N\geqq 1)$, we have an equation of the form $dW/dx=i\lambda x^{\frac{m}{2}}(\Lambda W-W\Lambda)+\lambda^{-1}(\tilde{C}W-W\tilde{D})+O(|\lambda|^{-N})$. Hence, by virtue of the definition of $p_j(\lambda)$ and the boundedness of the matrix W, we have an inequality $||W(x,\lambda)||\leqq M_N|\lambda|^{-N}$ in the region (7.1), where M_N is a suitable positive constant. Since $\lambda^{1-2\nu}\Psi W\Psi^{-1}=\lambda^{1-2\nu}\Psi Q\Psi^{-1}-P^{(N)}$, we also have another inequality

$$(9.1)\qquad ||\lambda^{1-2\nu}\Psi Q\Psi^{-1}-P^{(N)}||\leqq M_N'|\lambda|^{2-4\nu-N}$$

in the same region, with another positive constant M_N'.

We note further that the matrix $\lambda^{1-2\nu}\Psi Q\Psi^{-1}$ and the formal matrix P satisfy the same equation. Our theorem then follows immediately.

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