84. On Rings of Real Valued Continuous Functions

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E. Hewitt [3] proved several interesting and profound theorems on rings of real valued continuous functions, and threw a new light on the theory of completely regular spaces. We have aimed at the characterization of topology by the properties of function rings, and perceived that some of the algebraic techniques in the study of function rings might serve as a powerful one to solve topological problems. In the following, we shall limit ourselves to consideration of completely regular spaces. In this case, the rings of all real valued continuous functions are large enough to describe the topology of base spaces. As to the notations and terminologies, we shall use those of E. Hewitt [3].

1. In the first place, we shall give an algebraic proof of Theorem 3, which has already been proved by T. Shirota [6], using the notion of normal covering due to Tukey. At the same time, we shall obtain some results concerning the function ring on a subspace of completely regular space.

Let Y be any subspace of X, and let $C(X)$ and $C(Y)$ be the rings of all real valued continuous functions on X and Y respectively. Let $C'(Y)$ be the subring of $C(Y)$ consisting of all the functions which can be extended over X.

THEOREM 1. Let $\mathfrak{M}$ be any free maximal ideal in $C(Y)$ and let $\mathfrak{M}'=\mathfrak{M}\frown C'(Y)$. Then $\mathfrak{M}'$ is a free ideal in $C'(Y)$ (not necessarily maximal). If $\mathfrak{M}$ is real, then $\mathfrak{M}'$ is maximal and real.

Proof. Suppose that $\mathfrak{M}'$ is not free, then there is a point $p\in Y$ such that $\varphi(p)=0$ for every $\varphi\in\mathfrak{M}'$. Since $\mathfrak{M}$ is free, there is a function $f\in\mathfrak{M}$ such that $f(p)=1$. Let $U(p)=\{q\in Y;\ f(q)>\frac{1}{2}\}$ and let $V(p)$ be an open set of X such that $V(p)\frown Y\subset U(p)$. Since X is completely regular, there is a function $h\in C(X)$ such that $h(p)=1$ and $h(q)=0$ for every $q\notin V(p)$. Let h' be the restriction of h on Y, then it is clear that $h'\notin\mathfrak{M}$. Since $\mathfrak{M}$ is maximal, there is a function $g\in\mathfrak{M}$ such that $Z(g)\frown Z(h')=\phi$. Obviously $Z(f)\subset Z(h')$, and it follows that $Z(f^2+g^2)=Z(f)\frown Z(g)=\phi$, which is a contradiction, since $f\in\mathfrak{M}$ and $g\in\mathfrak{M}$. Moreover, if $\mathfrak{M}$ is real then for every $f'\in C'(Y)$, there is a real number a such that $f'-a\in\mathfrak{M}$. Since $a\in C'(Y)$, it follows that $f'-a\in\mathfrak{M}'$, which implies that $\mathfrak{M}'$ is maximal and real.

THEOREM 2. Let Y be a closed subspace of X. Then there is a homomorphism η of $C(X)$ onto $C'(Y)$, the kernel of which is the ideal

$\mathfrak{A}$ consisting of all the functions vanishing on Y. Let $\mathfrak{M}$ be a maximal ideal in $C(X)$, then $\mathfrak{M}'=\eta(\mathfrak{M})$ is also a maximal ideal in $C'(Y)$ if and only if $\mathfrak{M}$ contains $\mathfrak{A}$. In this way, maximal ideals of $C'(Y)$ and those of $C(X)$ containing $\mathfrak{A}$ are in one to one correspondence. Moreover, $\mathfrak{M}'$ is free if and only if $\mathfrak{M}$ is free, and $\mathfrak{M}'$ is real if and only if $\mathfrak{M}$ is real.

THEOREM 3. Every closed subspace of a Q-space is also a Q-space.

Proof. Let Y be a closed subspace of a Q-space X. Suppose that Y is not a Q-space, then there is at least one real free maximal ideal in $C(Y)$. Hence, there is a real free maximal ideal in $C'(Y)$ by Theorem 1 and therefore in $C(X)$ by Theorem 2, which contradicts the assumption that X is a Q-space.

2. This section is devoted to the characterization of topology in terms of free ideals.

THEOREM 4. A space is locally compact if and only if the intersection of all free ideals is also a free ideal.

Proof. Let $\mathfrak{A}$ be the intersection of all free maximal ideals in $C(X)$. Let g be an element of $C(X)$ such that $Z(g)\frown Z(f)=\phi$ for some $f\in\mathfrak{A}$, then $Z(g)$ is compact by Theorems 1 and 2. If $\mathfrak{A}$ is free, then there is, for every point $p\in X$, an element $f\in\mathfrak{A}$ such that $f(p)=1$. Put $g=\frac{1}{2}-\min[\frac{1}{2},f]$, then $Z(g)$ is a compact neighborhood of p. Conversely, if X is locally compact, then there is a free ideal $\mathfrak{A}_0$ consisting of all the functions with compact carrier. It is clear that $\mathfrak{A}_0$ is contained in every free ideal, which completes the proof.

COROLLARY. If the intersection of all free maximal ideals is free, then the intersection of all free ideals is also free.

A free ideal $\mathfrak{A}$ is said to be locally finite (star finite) if there is a locally finite partition of unity $\sum\varphi_\lambda=1$, such that $\varphi_\lambda\in\mathfrak{A}$, and $\{Z^c(\varphi_\lambda)\}$ is a locally infinite (star finite) covering of X.

THEOREM 5. A space is paracompact if and only if every free ideal is locally finite.

Proof. Let $\{U_\alpha\}$ be any open covering of X. Let $\mathfrak{F}$ be the set of all the functions with a carrier contained in some U_α, and let $\mathfrak{A}$ be the ideal generated by the elements of $\mathfrak{F}$. Then, by the assumption of the theorem, we have a locally finite partition of unity $\sum\varphi_\lambda=1$, where $\varphi_\lambda\in\mathfrak{A}$. Since each φ_λ can be represented as a finite sum of elements of $\mathfrak{F}$, we can easily obtain a locally finite refinement of $\{U_\alpha\}$. The converse follows by virtue of the well-known theorem due to Dieudonné [1, Theorem 6] and the fact that if $Z(g)\supset Z(f)$ for some $f\in\mathfrak{A}$ then there is a function $g'\in\mathfrak{A}$ such that $Z(g')=Z(g)$.

THEOREM 6. Let X be a completely regular space and let X_0 be the discrete space of the same cardinal number with that of connected components of X. Then X is a Q-space if and only if every free maximal ideal is star finite (provided that X_0 is a Q-space).

Proof. For every hyper-real ideal $\mathfrak{M}$, there is a function $h \in C(X)$ such that $P(h-a) \in Z(\mathfrak{M})$ for every real number a. Let $g_n = h'_n \cdot h''_n$ where $h'_n = (n+1) - \min[h, n+1]$, $h''_n = \max[h, n-2] - (n-2)$. Then it is clear that $g_n \in \mathfrak{A}$ and that $\{Z^c(g_n)\}$ is a star finite covering of X. If X is connected, then the converse follows immediately from the fact that a star finite covering of a connected space is countable. If X can be decomposed into noncountable number of connected components, we have two cases to be considered. Let $\mathfrak{M}$ be a free maximal ideal, and $\mathfrak{A}_\alpha$ be the ideal consisting of all the functions vanishing on X_α, where X_α is a connected component of X.

Case 1. $\mathfrak{M} \supset \mathfrak{A}_\alpha$ for some α. It is clear that there is only one such α. Let f_α be the function such that $f_\alpha = 0$ on X_α and $f_\alpha = 1$ on $\bigcup_{\beta \neq \alpha} X_\beta$, then f_α belongs to $\mathfrak{M}$, and it follows that $\mathfrak{M}$ is hyper-real.

Case 2. $\mathfrak{M} \not\supset \mathfrak{A}_\alpha$ for every α. Let g_α be the function such that $g_\alpha = 1$ on X_α and $g_\alpha = 0$ on $\bigcup_{\beta \neq \alpha} X_\beta$. By Theorem 2, it may be concluded that $g_\alpha \in \mathfrak{M}$ for every α, since $Z(g) \supset Z(f)$ for some $f \in \mathfrak{M}$ implies $g \in \mathfrak{M}$. Let $\varphi_1 = \sum_{\alpha \in A} g_\alpha$ be any partial sum of g_α's, and let $\varphi_2 = 1 - \varphi_1 = \sum_{\beta \notin A} g_\beta$, then clearly $\varphi_1 \in \mathfrak{M}$ or $\varphi_2 \in \mathfrak{M}$, and it follows that the subfamily $Z_0(\mathfrak{M}) = \{\bigcup_{\beta \notin A} X_\beta;\ \sum_{\alpha \in A} g_\alpha \in \mathfrak{M}\}$ in $Z_0(X) = \{\bigcup_{\alpha \in A} X_\alpha\}$ is maximal with respect to the finite intersection property. Therefore, (from our assumption), it may be seen that $Z_0(\mathfrak{M})$ contains a countable subfamily with total intersection void, and $\mathfrak{M}$ is accordingly hyper-real.

Note. The assumption concerning the cardinal number of connected components of X should be imposed on the above theorem, under the present situation. In view of the theorem due to L. Nachbin [5] and T. Shirota [7] which states that $C(X)$ is "bornologique" with respect to the compact-open topology if and only if X is a Q-space, it is clear that the followings are equivalent.

1) Every discrete space is a Q-space.

2) Every product space $\prod_\lambda R_\lambda$ of real number spaces is "bornologique". (It should also be noted that those are equivalent to Ulam's problem.)

Comparing conditions of Theorems 5 and 6, the following questions arise. What is the space satisfying each one of the following conditions?

a) Every free ideal is star finite.

b) Every free maximal ideal is locally finite.

THEOREM 7. A space is the topological sum of Lindelöf spaces if and only if every free ideal is star finite.

Proof. The proof may easily be obtained in the same way as in the proof of Theorem 5, in virtue of the results due to K. Morita [4].

It is well known that paracompact spaces are topologically complete and so are Q-spaces. Are the spaces satisfying condition b) topologically complete? Conversely, does every topologically complete space satisfy condition b)?

THEOREM 8. A space is topologically complete if and only if every free maximal ideal is locally finite.

To prove that the condition is necessary, a lemma should be prepared.

LEMMA. Let $\mathfrak{U}=\{V_\alpha\}$ be a uniformity for X, where each V_α may be assumed to be open and symmetric. If the uniform space $(X,\mathfrak{U})$ is complete, then for every $\mathfrak{M}\in\beta X$, corresponding to a free maximal ideal, there is a $V\in\mathfrak{U}$ such that $\widetilde{V}(\mathfrak{M})\frown X=\phi$, where $\widetilde{V}$ is the interior of the closure of V taken in $\beta X\times\beta X$.

Proof. If this is not the case, then there is a free maximal ideal $\mathfrak{M}$ such that $\widetilde{V}_\alpha(\mathfrak{M})\frown X=C_\alpha\neq\phi$, for every α. Then $\{C_\alpha\}$ is a Cauchy filter. This follows easily from the observation: If $V_\beta\cdot V_\beta\subset V_\alpha$, then $\widetilde{V}_\beta\cdot\widetilde{V}_\beta\subset\widetilde{V}_\alpha$, since X is dense in βX. Consider the filter $[F_{\lambda,\alpha}]=\{U_\lambda(\mathfrak{M})\frown C_\alpha\}$, where $U_\lambda(\mathfrak{M})$ is a neighborhood of $\mathfrak{M}$ in βX, then it is clear that $(X,\mathfrak{U})$ is not complete.

Proof of the theorem. Let X be complete relative to the uniformity $\mathfrak{U}$, then for every free maximal ideal $\mathfrak{M}_\nu$ there is a $V_\nu\in\mathfrak{U}$ as in the above lemma. Let d_ν be the pseudo-metric defined by V_ν, in a usual manner, such that $d_\nu(p,q)=1$ for every $(p,q)\notin V_\nu$. Let $X'_\nu=X/\mathfrak{R}_\nu$ be the quotient space defined by the relation $\mathfrak{R}_\nu=\{(p,q)\in X\times X;\ d_\nu(p,q)=0\}$, then X'_ν is metrizable, hence is paracompact. Let $p_\nu: X\to X'_\nu$ be the projection. Now, let $\{U(p)\}$ be the covering of X where $U(p)=\left\{q\in X;\ d_\nu(p,q)<\frac{1}{2^2}\right\}$. Then $\{U(p)\}$ has an open locally finite refinement $\{U_\lambda\}$, where $U_\lambda=p_\nu^{-1}(U'_\lambda)$, and $\{U'_\lambda\}$ is an open locally finite refinement of $\{p_\nu[U(p)]\}$. Moreover, we have a refinement $\{W_\lambda\}$ of $\{U_\lambda\}$, with the same indices λ, such that $\overline{W}_\lambda\subset U_\lambda$, and also the function f_λ such that $f_\lambda=1$ on $\overline{W}_\lambda$ and $f_\lambda=0$ outside of U_λ. In view of the definition of V_ν, it follows that $\overline{U}_\lambda\not\ni\mathfrak{M}_\nu$, where $\overline{U}_\lambda$ is the closure of U_λ taken in βX. Let $g_\lambda=\max[f_\lambda,\frac{1}{2}]-\frac{1}{2}$, then it is clear that $g_\lambda\in\mathfrak{M}$, and that $\{Z^c(g_\lambda)\}$ is a locally finite covering of X. Thus the necessity of the condition is clear. Conversely, if every free maximal ideal $\mathfrak{M}_\nu$ has a locally finite partition of unity $\sum\varphi_\lambda^{(\nu)}=1$, then we can define a pseudo-metric d_ν, corresponding to each $\mathfrak{M}_\nu$, as follows: $d_\nu(p,q)=\sum|\varphi_\lambda^{(\nu)}(p)-\varphi_\lambda^{(\nu)}(q)|$. Let $V_{\nu,n}=\left\{(p,q)\in X\times X;\ d_\nu(p,q)<\frac{1}{2^n}\right\}$, then the weakest uniformity $\mathfrak{U}_0$ for X and the family $\{V_{\nu,n}\}$, where ν,n run over all possible indices, generate a uniformity $\mathfrak{U}$. We shall show that the uniform space $(X, \mathfrak{U})$ is complete. Let $\{C_\alpha\}$ be any Cauchy filter relative to the uniformity $\mathfrak{U}$. Let C be an element of $\{C_\alpha\}$ such that $C\times C \subset V_{\nu,2}$, and take one point p from C, then all but a finite number of $\varphi_\lambda^{(\nu)}$'s vanish at p. Let φ_ν be the sum of all those $\varphi_\lambda^{(\nu)}$'s, which do not vanish at p, then it is clear that $\varphi_\nu \in \mathfrak{M}_\nu$, and that $\varphi_\nu(q) > \frac{1}{2}$ for every $q \in C$. Considering a neighborhood $\mathcal{V}_{\psi_\nu}$ of $\mathfrak{M}_\nu$ in βX, where $\psi_\nu = \max[\frac{1}{4}-\varphi_\nu, 0]$, we have $C \frown \mathcal{V}_{\psi_\nu} = \phi$. This means that $\mathfrak{M}_\nu$ is not a cluster point of $\{C_\alpha\}$. Consequently, it follows that no point in βX corresponding to a free maximal ideal is a cluster point of $\{C_\alpha\}$. Thus the proof is completed, since $\{C_\alpha\}$ has a cluster point in βX.

COROLLARY. If X is topologically complete, then every closed subset Y, on which every element of $C(X)$ is bounded, must be compact. In particular, every pseudo-compact topologically complete space is compact.

Proof. If Y is not compact, then there is a free maximal ideal in $C(Y)$, hence in $C'(Y)$ by Theorem 1 and hence in $C(X)$ by Theorem 2. Taking the associated locally finite partition of unity $\sum \varphi_\lambda = 1$, and choosing a countable number (infinite) of φ_λ's, such that $Z^c(\varphi_\lambda) \frown Y \neq \phi$, we have an unbounded function $\varphi = \sum_{k=1}^{\infty} a_k \varphi_k$, where a_k's are suitably chosen constants. Clearly φ is continuous, since $\{Z^c(\varphi_\lambda)\}$ is a locally finite covering of X.

From the proof of Theorem 8, we have the following

THEOREM 9. A uniform space $(X, \mathfrak{U})$ is complete if and only if for every free maximal ideal $\mathfrak{M}$, there is a $V \in \mathfrak{U}$ such that; if $C\times C \subset V$ then $\bar{C} \not\ni \mathfrak{M}$ (in βX).

Now let us consider the relationship between free ideals and uniformity for the base space X.

THEOREM 10. Let $\mathfrak{A}$ be any free ideal, then there is a uniformity $\mathfrak{U}$ for X relative to which $Z(\mathfrak{A})$ is a Cauchy filter.

Proof. Let $V_{f,n} = \left\{(p,q) \in X\times X;\ |f(p)-f(q)| < \frac{1}{2^n},\ f \in \mathfrak{A}\right\}$. Then $\mathfrak{U} = \{V_{f,n}\}$ is a desired uniformity.

THEOREM 11. X is a Q-space if and only if it is complete relative to the weakest uniform structure with respect to which every element of $C(X)$ is uniformly continuous.

Proof. Let $\{C_\alpha\}$ be a Cauchy filter. Let $\mathfrak{M}$ be the ideal consisting of all the functions $f \in C(X)$ such that, for every $\varepsilon > 0$ there is a C_α on which $|f| < \varepsilon$. Then $\mathfrak{M}$ is a real maximal ideal in $C(X)$. Consequently, the completion of X relative to the uniform structure is νX. This proves the proposition.

THEOREM 12. X has unique uniform structure if and only if $C(X)$

has at most one free maximal ideal. Moreover the unique free maximal ideal is real, in this case, and therefore X must be pseudo-compact.

Proof. The necessity of the condition is almost evident in view of Theorem 10, and the last statement follows from Theorem 11 and the fact that, X is pseudo-compact if and only if every maximal ideal is real. Conversely, if X has only one free maximal ideal, then it follows immediately that out of the two normally separable closed subsets, one, at least, must be compact. This completes the proof (see R. Doss [2]).

COROLLARY 1. If $C(X)$ has only one free maximal ideal $\mathfrak{M}$, then $\mathfrak{M}$ is real.

COROLLARY 2. If a normal space has unique uniform structure, then it must be compact.

Proof. This is an immediate consequence of the fact that every normal pseudo-compact space is compact.

In view of the above propositions, it may be stated that compact, Lindelöf, paracompact, normal, Q, and topologically complete spaces are distinguished from one another, with the mode of existence of unbounded functions.

References

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