94. *Ideals in Non-commutative Lattices*

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§ 1. We published in 1953 a résumé of the theory of non-commutative lattices in C. R. Acad. Sci., Paris [11]. After this, we have received communications from Dr. F. Klein-Barmen and also from Prof. Dr. Pascual Jordan that Prof. P. Jordan with collaborators, Dr. E. Witt and Dr. W. Böge, had been constructing the theory of non-commutative lattices, independently of us, for the sake of applications in "theoretical physics" [3-6], and also independently, F. Klein published some excellent and interesting works on the similar articles [7-10].

Here, we shall make a survey of ideal theory in non-commutative lattices, from which the structure of some kinds of non-commutative lattices (normal and regular type) is decided (see § 3). This paper is also a résumé; and a full note, with complete proofs of [1], (titled "Theorie der nichtkommutativen Verbände I-II") will appear elsewhere.*)

§ 2. Let $\mathfrak{A}$ be an algebraic system with binary operators $*$ and $\circ$, both of which are associative and idempotent, but not necessarily commutative. If an order $<$ in $\mathfrak{A}$, i.e. i) $x<x$, ii) $x<y, y<x \rightarrow x=y$, iii) $x<y, y<z \rightarrow x<z$ for $x, y, z \in \mathfrak{A}$, satisfies a further condition: for any $a \in \mathfrak{A}$,

(2.1) $$x<y \rightarrow a*x<a*y \quad (\text{or } x*a<y*a),$$

then it is called left (resp. right) $*$-order of $\mathfrak{A}$. And if a left (or right) $*$-order $<$ of $\mathfrak{A}$ fulfils

(2.2) $$x<x*a \ (\text{resp. } x<a*x) \quad \text{for any } x, a \in \mathfrak{A},$$

then such $<$ is called a left (resp. right) *L-$*$-order* of $\mathfrak{A}$. Similarly, a left (or right) *L-$\circ$-order* of $\mathfrak{A}$ is defined.

Theorem 1. *In order that $\mathfrak{A}$ admit at least one left or right L-$*$-order, it is necessary and sufficient that the following equality be kept in $\mathfrak{A}$;*

(2.3) $$\alpha) \ x*a=x*a*x \quad \textit{resp.} \quad \beta) \ a*x=x*a*x.$$

An order $<$ (or $\prec$) is called *stronger* (resp. *weaker*) than $\prec$ (resp. $<$) if $a\prec b$ yields $a<b$: then

Theorem 2. *Suppose that $\mathfrak{A}$ satisfies the condition α) (or β)) in* (2.3) *above: Then*

I) *The order in $\mathfrak{A}$ defined by $a<b$ if and only if $b=a*x$ (resp.*

*) I express my hearty thanks to Dr. F. Klein-Barmen and also to Prof. Dr. Van der Waerden and his assistant Dr. R. Fischer for their precious advices and the precise examination of my theory.

$x*a$) for a suitable $x \in \mathfrak{A}$ is a left (resp. right) L-$*$-order of $\mathfrak{A}$ and it is the weakest one, that is, any possible L-$*$-order of $\mathfrak{A}$ is stronger than it.

II) *The order defined by $a<b$ if and only if $b=a*b$ (resp. $=b*a$) is an L-$*$-order in $\mathfrak{A}$. When $a<b$ for this order, we have*

$$b=a*b=b*a. \tag{2.4}$$

III) *The order defined by $a<b$ whenever $b=b*a$ (resp. $=a*b$) is an L-$*$-order of $\mathfrak{A}$, if and only if $\mathfrak{A}$ is commutative. In the case, this is the only L-$*$-order of $\mathfrak{A}$.*

The orders defined in I), II), and III) above are called the *weakest*, *regular*, and the *strongest* L-$*$-order of $\mathfrak{A}$ respectively. These are all the same for L-$\circ$-order.

If the L-$*$-order $<$ and L-$\circ$-order $\prec$ of $\mathfrak{A}$ are mutually reciprocal, i.e. $a<b \rightleftarrows b\prec a$, then $\mathfrak{A}$ is called a *quasi-lattice*, and further if these two reciprocal orders are both regular, $\mathfrak{A}$ is called a *regular quasi-lattice*. Moreover, $\mathfrak{A}$ is said to be of (l, l), (l, r), (r, l) or (r, r) *type* according to what type (left or right) of L-$*$- and L-$\circ$-orders of $\mathfrak{A}$ to be adopted.

We notice that if $\mathfrak{A}$ is a regular quasi-lattice, $\mathfrak{A}$ is determined only by its algebraic structure without considering any order structure, just as in usual (i.e. commutative) lattices which are defined purely algebraically by the method of F. Klein or of G. Birkhoff.[1]

Theorem 3. *In a quasi-lattice $\mathfrak{A}$ of (l, l), (l, r), (r, l), or (r, r) type, we have respectively*

i) $a*(a\circ x)=a=a\circ(a*x)$,

ii) $a*(x\circ a)=a=(a*x)\circ a$,

iii) $(a\circ x)*a=a=a\circ(x*a)$,

iv) $(x\circ a)*a=a=(x*a)\circ a$.

These four equalities are called the *absorption laws* of the respective types.

§3. Hereafter, let $\mathfrak{A}$ be a regular quasi-lattice and $\mathfrak{M}$ a subsystem of $\mathfrak{A}$ (that is, $\mathfrak{M}$ is closed with respect to $*$ and $\circ$). If for every $a\in\mathfrak{M}$ and $p\in\mathfrak{A}$, we have

i) $a*p$ and $a\circ p\in\mathfrak{M}$, (l, l) type,

ii) $a*p$ and $p\circ a\in\mathfrak{M}$, (l, r) type,

iii) $p*a$ and $a\circ p\in\mathfrak{M}$, (r, l) type,

iv) $p*a$ and $p\circ a\in\mathfrak{M}$, (r, r) type,

then $\mathfrak{M}$ is called an *ideal* of $\mathfrak{A}$ in respective types.[2]

A usual (i.e. commutative) lattice $\mathfrak{B}$ is *simple*, that is, $\mathfrak{B}$ has a single ideal (in the present sense) which coincides with $\mathfrak{B}$ itself.

Lemma 1. *The non-void intersection or union of any numbers of*

1) See e.g., F. Klein: Grundzüge der Theorie der Verbände, Math. Ann., **111** (1935) and G. Birkhoff [1], J. Dubreil-L. Lesieur-R. Croisot [2].

2) This concept of ideal is different from that of usual lattice theory; cf. [1, 2].

ideals of a certain common type forms also an ideal of the same type.

We say that an ideal $\mathfrak{M}$ which contains no other ideal than $\mathfrak{M}$ itself is *a minimal ideal*; then we have

Theorem 4. *For each $x \in \mathfrak{A}$, there exists a minimal ideal $\mathfrak{M}$ which contains x.*

At first, if we have $y < x$ and $y < z$, or otherwise $x < y$ and $z < y$, we say that z is *order-connected* with x. Now, we shall prove the theorem in the case of (l, l) type. Denote by $\mathfrak{M}_x$ the collection of all order-connected elements with x; it is clear that $\mathfrak{M}_x$ forms an ideal. Suppose that there were an ideal $\mathfrak{N}$ contained in $\mathfrak{M}_x$; for an arbitrary $z \in \mathfrak{N}$, we find such a y that either $x < y$ and $z < y$ or $y < x$ and $y < z$. In the first case, $y = z*y \in \mathfrak{N}$ and $x = y \circ x \in \mathfrak{N}$, while in the latter $y = z \circ y \in \mathfrak{N}$ and hence $x = y*x \in \mathfrak{N}$. For any $a \in \mathfrak{M}_x$, we can conclude by the similar way that $a \in \mathfrak{N}$, since a is order-connected with $x \in \mathfrak{N}$; thus, $\mathfrak{N} = \mathfrak{M}_x$. For $\mathfrak{A}$ of any other type, it is the same at all.

On account of Lemma 1, we have

Theorem 5. *A regular quasi-lattice $\mathfrak{A}$ is decomposed in the form:*

(3.1) $$\mathfrak{A} = \sum \oplus_{\lambda \in \Lambda} \mathfrak{M}_\lambda,$$

where $\mathfrak{M}_\lambda$ is a minimal ideal for each $\lambda \in \Lambda$, and $\mathfrak{M}_\lambda \frown \mathfrak{M}_{\lambda'}$ is empty for $\lambda \neq \lambda'$.

Now, let $\mathfrak{A}$ be of (l, l) type. When $x = x*a$, we set $a < x(*)$ and if $x < y(*)$ and simultaneously $y < x(*)$ we define $x \sim y(*)$. Then, this is a congruence relation.[3] Moreover, if $x < y$ for the (left) regular L-$*$-order, then $x < y(*)$ and, for any $t \in \mathfrak{A}$, from $x < y(*)$ follows $t*x < t*y(*)$. Similarly, we define $x < y(\circ)$ and $x \sim y(\circ)$.

Lemma 2. *$x*y \sim y*x(*)$ and $x \circ y \sim y \circ x(\circ)$.*

By these congruences, we obtain the classifications of $\mathfrak{A}$ with respect to $*$ and $\circ$, denoted by $[\mathfrak{A}]^*$ and $[\mathfrak{A}]^\circ$ respectively; these form two residual algebraic systems of $\mathfrak{A}$ with respect to $*$ and resp. $\circ$, which are commutative by Lemma 2 and ordered by $<(*)$ resp. $<(\circ)$.

Theorem 6. *Let $\mathfrak{A}$ be a regular quasi-lattice; $[\mathfrak{A}]^*([\mathfrak{A}]^\circ)$ is $*$- (resp. $\circ$-) homomorphic to $\mathfrak{A}$.*

However, two orders $<(*)$ and $<(\circ)$ are not necessarily reciprocal. In order that these are mutually reciprocal, it is necessary and sufficient that

N_1) $$a*(x \circ a) = a = a \circ (x*a),$$

and if this equality is fulfiled, we have $[\mathfrak{A}]^* = [\mathfrak{A}]^\circ$ (put $= [\mathfrak{A}]$), which forms a usual lattice.[4]

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3) That is, i) $x \sim x(*)$, ii) $x \sim y(*) \rightarrow y \sim x(*)$, iii) $x \sim y(*)$, $y \sim z(*) \rightarrow x \sim z(*)$.

4) N_1) should be replaced in (l, r), (r, l), (r, r) types by

N_2) $a*(a \circ x) = a = (x*a) \circ a$, (l, r) type,

N_3) $(x \circ a)*a = a = a \circ (x*a)$, (r, l) type,

N_4) $(a \circ x)*a = a = (a*x) \circ a$, (r, r) type.

Now, we call the following conditions *modular law* in respective types: for $a<c$, and for $b\in\mathfrak{A}$,

M_1)	$a*(b\circ c)<c\circ(b*a)$,	(l, l) type,
M_2)	$a*(c\circ b)<(b*a)\circ c$,	(l, r) type,
M_3)	$(b\circ c)*a<c\circ(a*b)$,	(r, l) type,
M_4)	$(c\circ b)*a<(a*b)\circ c$,	(r, r) type.

If M_i) is fulfiled, then N_i) is guaranteed for each $i=1, 2, 3, 4$. A regular quasi-lattice $\mathfrak{A}$ (or an ideal $\mathfrak{M}$ in it) is called *normal* if for any pair a, c in $\mathfrak{A}$ (or $\mathfrak{M}$) with $a<c$ the modular law is satisfied, and otherwise *singular*.[5] Then, we establish:

Theorem 7. *A minimal ideal $\mathfrak{M}$ of a normal regular quasi-lattice $\mathfrak{A}$ forms a usual lattice to which $[\mathfrak{A}]$ is lattice-homomorphic.*

Indeed, for any $x\in\mathfrak{A}$, $x=x*(p\circ x)\sim(p\circ x)*x\in\mathfrak{M}$ for $p\in\mathfrak{M}$.

Theorem 8. *A regular quasi-lattice $\mathfrak{A}$ is decomposed in direct sum of normal $\mathfrak{A}^1$ and singular $\mathfrak{A}^2$,*

$$\mathfrak{A} = \mathfrak{A}^1 \oplus \mathfrak{A}^2, \tag{3.2}$$

with $\mathfrak{A}^i=\sum\oplus_{\lambda\in\Lambda_i}\mathfrak{M}^i_\lambda$ $(i=1, 2)$, $\mathfrak{M}^1_\lambda(\mathfrak{M}^2_\lambda)$ being a normal (normal or singular) minimal ideal; moreover $\mathfrak{M}^1_\lambda(\lambda\in\Lambda_1)$ are usual lattices, to which $[\mathfrak{A}^1]$ is lattice-homomorphic.

$$\mathfrak{M}^1_\lambda \cong [\mathfrak{M}^1_\lambda] \approx [\mathfrak{A}^1]. \tag{3.3}$$

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5) Some of the most simple models of singular regular quasi-lattice of (l, l) type are written in the following schemata i) and ii):

$(A\to B$ means $A<B)$.