91. On Zeta-Functions and L-Series of Algebraic Varieties. II

By Makoto ISHIDA
Mathematical Institute, University of Tokyo
(Comm. by Z. SUETUNA, M.J.A., July 12, 1958)

Here I shall give some supplementary results to my previous paper [1].

Let k be a finite field with q elements. Then, for an abelian variety B defined over k, π_B denotes the endomorphism of B such that $\pi_B(b)=b^q$ for all points b on B and M_l denotes the l-adic representation of the ring of endomorphisms of B for a (fixed) rational prime l different from the characteristic of k.

1. Let A/V be a Galois (not necessarily unramified) covering defined over k, with group G and of degree n, where A is an abelian variety and V is a normal projective variety (both defined over k); let r be the dimensions of A and V. Then, in this section, we shall explain the behaviors of the zeta-function $Z(u, V)$ of V and the L-series $L(u, \chi, A/V)$ of A/V over k in the circle $|u|<q^{-(r-3/2)}$ and $|u|<q^{-(r-1)}$ respectively.

Now let η_σ be the automorphism of A induced by an element σ of G and let $\pi=\pi_A$. Then $Z(u, V)$ and $L(u, \chi, A/V)$ are given by the following logarithmic derivatives:

$$d/du\cdot\log Z(u, V)=\textstyle\sum_{m=1}^{\infty}\{1/n\cdot\sum_{\sigma\in G}\det M_l(\pi^m-\eta_\sigma)\}u^{m-1},$$

$$d/du\cdot\log L(u, \chi, A/V)=\textstyle\sum_{m=1}^{\infty}\{1/n\cdot\sum_{\sigma\in G}\det M_l(\pi^m-\eta_\sigma)\chi(\sigma)\}u^{m-1}.$$

First we shall calculate $\det M_l(\pi^m-\eta_\sigma)$. If we transform the representation M_l of G (i.e. the restriction of M_l to G such that $M_l(\sigma)=M_l(\eta_\sigma)$) into the following form:

$$(*)\qquad M_l\,|\,G=\begin{pmatrix} E_{d_1}\times 1 & & & 0 \\ & E_{d_\chi}\times F_\chi & & \\ & & E_{d_{\chi'}}\times F_{\chi'} & \\ 0 & & & \ddots \end{pmatrix},^{1)}$$

where $1, F_\chi, F_{\chi'},\cdots$ are non-equivalent irreducible representations of G with characters $1, \chi, \chi',\cdots$ respectively, then, as $\pi\eta_\sigma=\eta_\sigma\pi$ for every σ in G, $M_l(\pi)$ must be transformed into the following form simultaneously:

$$(**)\qquad M_l(\pi)=\begin{pmatrix} (\pi_{ij}^{(1)})\times E_{f_1} & & & 0 \\ & (\pi_{ij}^{(\chi)})\times E_{f_\chi} & & \\ & & (\pi_{ij}^{(\chi')})\times E_{f_{\chi'}} & \\ 0 & & & \ddots \end{pmatrix},^{1)}$$

where $(\pi_{ij}^{(\chi)})$ is a matrix of degree d_χ and f_χ is the degree of F_χ.

1) In the following, the matrices $E_{d_1}\times 1$ and $(\pi_{ij}^{(1)})\times E_{f_1}$ do not appear if $d_1=0$.

(In the above expressions, E_d means the unit matrix of degree d.) Hence we have

$$\det M_l(\pi^m-\eta_\sigma)=\Pi_\chi \det|(\pi_{ij}^{(\chi)})^m\times E_{f_\chi}-E_{d_\chi}\times F_\chi(\sigma)|.$$

For fixed F_χ and σ, let $\lambda_1(\sigma),\cdots,\lambda_{f_\chi}(\sigma)$ be the characteristic roots of $F_\chi(\sigma)$; and also let $\pi_1^{(\chi)},\cdots,\pi_{d_\chi}^{(\chi)}$ be those of $(\pi_{ij}^{(\chi)})$. We note that $\pi_i^{(\chi)}$, $1\leq i\leq d_\chi$, are of course the characteristic roots of $M_l(\pi)$ and so of absolute values $q^{1/2}$. Then, by a matrix of the form $P\times Q$ where P, Q are non-singular matrices of degrees d_χ, f_χ, we can transform $E_{d_\chi}\times F_\chi(\sigma)$ and $(\pi_{ij}^{(\chi)})^m\times E_{f_\chi}$ simultaneously into

$$E_{d_\chi}\times\begin{pmatrix}\lambda_1(\sigma) & & 0\\ & \ddots & \\ 0 & & \lambda_{f_\chi}(\sigma)\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}\pi_1^{(\chi)^m} & & 0\\ & \ddots & \\ * & & \pi_{d_\chi}^{(\chi)^m}\end{pmatrix}\times E_{f_\chi}$$

respectively. So we have

$$\begin{aligned}\det|(\pi_{ij}^{(\chi)})^m\times E_{f_\chi}-E_{d_\chi}\times F_\chi(\sigma)| &=\Pi_{i=1}^{d_\chi}\Pi_{j=1}^{f_\chi}(\pi_i^{(\chi)^m}-\lambda_j(\sigma))\\ &=Q_\chi^m-\textstyle\sum_i (Q_\chi\pi_i^{(\chi)^{-1}})^m\chi(\sigma)+\sum_{i\neq j}(Q_\chi\pi_i^{(\chi)^{-1}}\pi_j^{(\chi)^{-1}})^m\chi(\sigma)^2\\ &\quad+\textstyle\sum_i (Q_\chi\pi_i^{(\chi)^{-2}})^m\cdot 1/2\cdot\{\chi(\sigma)^2-\chi(\sigma^2)\}+O(q^{m(f_\chi d_\chi-3)/2}),\end{aligned}$$

where $Q_\chi=\det|(\pi_{ij}^{(\chi)})\times E_{f_\chi}|=(\pi_1^{(\chi)}\cdots\pi_{d_\chi}^{(\chi)})^{f_\chi}$.

Therefore we have

$$\begin{aligned}\det M_l(\pi^m-\eta_\sigma)&=q^{mr}-\textstyle\sum_\chi\sum_i(q^r\pi_i^{(\chi)^{-1}})^m\chi(\sigma)\\ &+\textstyle\sum_\chi\sum_{i\neq j}(q^r\pi_i^{(\chi)^{-1}}\pi_j^{(\chi)^{-1}})^m\chi(\sigma)^2+\sum_\chi\sum_i(q^r\pi_i^{(\chi)^{-2}})^m\cdot 1/2\cdot\{\chi(\sigma)^2-\chi(\sigma^2)\}\\ &+\textstyle\sum_{\chi,\chi':\chi\neq\chi'}\sum_{i,j}(q^r\pi_i^{(\chi)^{-1}}\pi_j^{(\chi')^{-1}})^m\chi(\sigma)\chi'(\sigma)+O(q^{m(r-3/2)}).\end{aligned}$$

Here we remark that, as the traces of the l-adic representations are rational numbers, the character of $M_l|G$ is rational; and so if χ appears in the character of $M_l|G$, then $\bar{\chi}$ also appears in it, where $\bar{\chi}(\sigma)=\chi(\sigma^{-1})$. Moreover, by the expressions (*) and (**), it is easily verified that the set $\{\pi_1^{(\chi)},\cdots,\pi_{d_\chi}^{(\chi)}\}=\{q\bar{\pi}_1^{(\chi)^{-1}},\cdots,q\bar{\pi}_{d_\chi}^{(\chi)^{-1}}\}$ is identical with the set $\{q\pi_1^{(\bar{\chi})^{-1}},\cdots,q\pi_{d_\chi}^{(\bar{\chi})^{-1}}\}$ completely.

Before stating the main results, we shall give three lemmas; except the last one, they are entirely of group-theoretical nature.

Lemma 1. *Let H be a finite group of order h and F_χ an irreducible representation of H with character χ and of degree f. Then, for any irreducible character χ' of H, we have*

$$\textstyle\sum_{\tau\in H}\chi'(\tau^{-1})F_\chi(\tau)=\begin{cases} h/f\cdot E_f, & \text{if } \chi=\chi',\\ 0, & \text{if } \chi\neq\chi'.\end{cases}$$

Proof. If we put $M_{\chi'}=\sum_{\tau\in H}\chi'(\tau^{-1})F_\chi(\tau)$, we have $F_\chi(\sigma)M_{\chi'}=M_{\chi'}F_\chi(\sigma)$ for every σ in H. Therefore, by Schur's lemma, we have $M_{\chi'}=c\cdot E_f$ and so $f\cdot c=TrM_{\chi'}=\sum_{\tau\in H}\chi'(\tau^{-1})\chi(\tau)$. Then our assertion is clear.

Lemma 2. *Let H be a finite group of order h and χ an irreducible character of H. Then we have $\sum_{\tau\in H}\{\chi(\tau)^2-\chi(\tau^2)\}=0$.*

Proof. Let $F:\tau\to F(\tau)=(a_{ij}(\tau))$ be the representation of H with character χ. Then $F^*:\tau\to F^*(\tau)=(a_{ij}^*(\tau))={}^tF(\tau^{-1})=(a_{ji}(\tau^{-1}))$ is also an irreducible representation of H and we have $\chi(\tau^2)=\sum_i a_{ii}(\tau^2)$ $=\sum_{i,j} a_{ij}(\tau)a_{ji}(\tau)=\sum_{i,j} a_{ij}(\tau)a_{ij}^*(\tau^{-1})$. Hence, by Schur [3], we have

$$\sum_{\tau\in H}\chi(\tau^2)=\begin{cases} 0, & \text{if } F \text{ and } F^* \text{ are not equivalent,} \\ h, & \text{if } F \text{ and } F^* \text{ are equivalent.}\end{cases}$$

On the other hand, if χ^* is the character of F^*, we have $\sum_\tau \chi(\tau)^2$ $=\sum_\tau \chi(\tau)\chi^*(\tau^{-1})$ and so

$$\sum_{\tau\in H}\chi(\tau)^2=\begin{cases} 0, & \text{if } F \text{ and } F^* \text{ are not equivalent,} \\ h, & \text{if } F \text{ and } F^* \text{ are equivalent.}\end{cases}$$

Lemma 3. *The Albanese variety $A(V)$ of V is isogenous to $\rho(A)$, where $\rho=\sum_{\sigma\in G}\eta_\sigma$, and $M_l(\pi_{A(V)})$ is equivalent to $(\pi_{ij}^{(1)})$ in $(**)$.*

Proof. The first assertion is proved similarly as in the proof of Theorem 3 in Ishida [1]. From $(*)$ we have, by Lemma 1,

$$M_l(\rho)=M_l(\textstyle\sum_\sigma \eta_\sigma)=\begin{pmatrix} n\cdot E_{d_1} & 0 \\ 0 & 0\end{pmatrix},$$

and so $M_l(\pi_{\rho(A)})(n\cdot E_{d_1}\ 0)=(n\cdot E_{d_1}\ 0)M_l(\pi_A)$. Hence $M_l(\pi_{\rho(A)})$ is equivalent to $(\pi_{ij}^{(1)})$ and the second assertion follows from the first.

Theorem 1. *Let $P(u)=\prod_{i=1}^{2g}(1-q^{r-1}\pi_i^* u)$,[2] where $\pi_1^*,\cdots,\pi_{2g}^*$ are the characteristic roots of $M_l(\pi_{A(V)})$ and g is the dimension of $A(V)$; let d_χ be the multiplicity of χ in the character of $M_l\,|\,G$. Then*

$$Z(u,V)\cdot(1-q^r u)/P(u)$$

has $\sum_{\chi:\chi=\bar\chi} 1/2\cdot d_\chi(d_\chi-1)+\sum_{\chi,\bar\chi:\chi\neq\bar\chi} d_\chi d_{\bar\chi}$ poles on the circle $|u|=q^{-(r-1)}$ and, except them, it has neither zero nor pole in the circle $|u|<q^{-(r-3/2)}$.

Proof. From the expressions of det $M_l(\pi^m-\eta_\sigma)$, using the orthogonal relation of group-characters and Lemma 2, we have

$$1/n\cdot\sum_{\sigma\in G}\det M_l(\pi^m-\eta_\sigma)=q^{mr}-\sum_{i=1}^{d_1}(q^r\pi_i^{(1)^{-1}})^m$$
$$+\sum_{\chi:\chi=\bar\chi}\sum_{i\neq j}(q^r\pi_i^{(\chi)^{-1}}\pi_j^{(\chi)^{-1}})^m+\sum_{\chi,\bar\chi:\chi\neq\bar\chi}\sum_{i,j}(q^r\pi_i^{(\chi)^{-1}}\pi_j^{(\bar\chi)^{-1}})^m$$
$$+O(q^{m(r-3/2)}).$$

Then the above remarks and Lemma 3 show that the set $\{\pi_1^*,\cdots,\pi_{2g}^*\}$ is identical with the set $\{\pi_1^{(1)},\cdots,\pi_{d_1}^{(1)}\}=\{q\pi_1^{(1)^{-1}},\cdots,q\pi_{d_1}^{(1)^{-1}}\}$ completely and $2g=d_1$. Therefore we have

$$d/du\cdot\log Z(u,V)+d/du\cdot\log(1-q^r u)-d/du\cdot\log P(u)$$
$$=\sum_{m=1}^{\infty}\{\sum_{\chi:\chi=\bar\chi}\sum_{i\neq j}+\sum_{\chi,\bar\chi:\chi\neq\bar\chi}\sum_{i,j}+C_m q^{m(r-3/2)}\}u^{m-1},$$

where C_m is a constant bounded in absolute value by a fixed constant C. Then all the assertions are easily verified.

Remark. In the case where $g\geq 1$ or G is abelian and, moreover, in many other cases, we can show that $u=q^{-(r-1)}$ is a pole of $Z(u,V)\cdot$ $(1-q^r u)/P(u)$ (on the circle $|u|=q^{-(r-1)}$).

Corollary. *Let $N(A,k_m)$ and $N(V,k_m)$ be the numbers of rational points of A and V over k_m, the (unique) extension over k of degree m. Then V is also an abelian variety over k if and only if we have*

$$N(A,k_m)-N(V,k_m)=O(q^{m(r-1)})$$

for all $m\geq 1$.

2) If d_1 (or d_χ)$=0$, we put, in the following, $P(u)$ (or $P_\chi(u)$)$=1$.

Proof. Clearly $N(A, k_m)-N(V, k_m)=O(q^{m(r-1)})$ is equivalent to the fact that $Z(u, A)/Z(u, V)$ has neither zero nor pole in the circle $|u|<q^{-(r-1)}$ and also that $d_1=2g=2r$ i.e. the degree of M_l. This equality holds if and only if $M_l\,|\,G=E_{2r}$, i.e. all $\eta_\sigma=1$. And this is clearly a necessary and sufficient condition for V to be an abelian variety over k.

Theorem 2. *Let $P_\chi(u)=\prod_{i=1}^{d_\chi}(1-q^{r-1}\pi_i^{(\chi)}u)$, where $\pi_1^{(\chi)},\cdots,\pi_{d_\chi}^{(\chi)}$ are the characteristic roots of $(\pi_{ij}^{(\chi)})$ and d_χ is the multiplicity of χ in the character of $M_l\,|\,G$. Then, for $\chi\neq 1$,*

$$L(u, \chi, A/V)/P_\chi(u)$$

has neither zero nor pole in the circle $|u|<q^{-(r-1)}$.

Proof. Similarly as in the proof of the preceding theorem, we have

$$1/n\cdot\sum_{\sigma\in G}\det M_l(\pi^m-\eta_\sigma)\chi(\sigma)$$
$$=-\sum_{i=1}^{d_\chi}(q^r\pi_i^{(\bar{\chi})^{-1}})^m+O(q^{m(r-1)})=-\sum_{i=1}^{d_\chi}(q^{r-1}\pi_i^{(\chi)})^m+O(q^{m(r-1)})$$

and so

$$d/du\cdot\log L(u, \chi, A/V)-d/du\cdot\log P_\chi(u)=\sum_{m=1}^{\infty}C_m^{(\chi)}q^{m(r-1)}u^{m-1},$$

where $C_m^{(\chi)}$ is a constant bounded in absolute value by a fixed constant $C^{(\chi)}$. Then all the assertions are easily verified.

2. Let U/V be a Galois (not necessarily unramified) covering, defined over k, with group G and of degree n, where U, V are non-singular projective varieties (both defined over k) of dimension r. Then, from the preceding theorems, we can give the following conjectural statement on the behaviors of the L-series $L(u, \chi, U/V)$ of U/V over k in the circle $|u|<q^{-(r-1)}$, which is equivalent to that given by Lang. (As for the definition of L-series in general cases, see Lang [2].) Let $A(U)$ be the Albanese variety of U. As k is a finite field, we may assume that $A(U)$ and the canonical map: $U\to A(U)$ are defined over k. Then every element σ in G induces an automorphism η_σ, defined over k, of $A(U)$ and $\pi_{A(U)}\eta_\sigma=\eta_\sigma\pi_{A(U)}$ for every σ in G; and so we have similarly as in 1:

$$M_l\,|\,G=\begin{pmatrix} E_{d_1}\times 1 & & & 0 \\ & E_{d_\chi}\times F_\chi & & \\ & & E_{d_{\chi'}}\times F_{\chi'} & \\ 0 & & & \ddots \end{pmatrix},\quad M_l(\pi_{A(U)})=\begin{pmatrix} (\pi_{ij}^{(1)})\times E_{f_1} & & & 0 \\ & (\pi_{ij}^{(\chi)})\times E_{f_\chi} & & \\ & & (\pi_{ij}^{(\chi')})\times E_{f_{\chi'}} & \\ 0 & & & \ddots \end{pmatrix},$$

where $1, F_\chi, F_{\chi'},\cdots$ are non-equivalent irreducible representations of G with characters $1, \chi, \chi',\cdots$ and of degrees $f_1=1, f_\chi, f_{\chi'},\cdots$. We put $P_\chi(u)=\prod_{i=1}^{d_\chi}(1-q^{r-1}\pi_i^{(\chi)}u)$, where $\pi_1^{(\chi)},\cdots,\pi_{d_\chi}^{(\chi)}$ are the characteristic roots of $(\pi_{ij}^{(\chi)})$ and $a_\chi=1$ if $\chi=1$ and $a_\chi=0$ if $\chi\neq 1$. Then, *for every χ* (not excluding $\chi=1$),

$$L(u, \chi, U/V)\cdot(1-q^r u)^{a_\chi}/P_\chi(u)$$

has neither zero nor pole in the circle $|u|<q^{-(r-1)}$. As, in this general

case, we can also prove that $\rho(A(U))$ is isogenous to the Albanese variety $A(V)$ of V where $\rho=\sum_{\sigma\in G}\eta_\sigma$, this conjecture complements those of Weil and Lang.

In the case where $U=\Gamma$, $V=\Gamma_0$ are non-singular complete curves defined over k, this conjecture is easily verified. In fact, by Weil [4, 5], we have

$$L(u,\chi,\Gamma/\Gamma_0)=\exp\left(-\int_0^u \Phi_\chi(u)\cdot du/u\right)\ (1-u)^{a\chi}(1-qu)^{a\chi},$$

where $\Phi_\chi(u)=1/f_\chi\cdot\sum_{m=1}^\infty \mathrm{Tr}\, M_l(\rho_\chi\pi^m)u^m$, $\rho_\chi=f_\chi/n\cdot\sum_{\sigma\in G}\chi(\sigma^{-1})\eta_\sigma$ and $\pi=\pi_{A(\Gamma)}$. Then by Lemma 1,

$$M_l(\rho_\chi\pi^m)=f_\chi/n\cdot M_l\left(\textstyle\sum_\sigma\chi(\sigma^{-1})\eta_\sigma\right)M_l(\pi^m)$$

$$=f_\chi/n\cdot\begin{pmatrix}0 & & \\ & n/f_\chi\cdot E_{d_\chi}\times E_{f_\chi} & \\ & & 0\end{pmatrix}\cdot M_l(\pi^m)=\begin{pmatrix}0 & & \\ & (\pi_{ij}^{(\chi)})^m\times E_{f_\chi} & \\ & & 0\end{pmatrix};^{3)}$$

and so we have $\mathrm{Tr}\, M_l(\rho_\chi\pi^m)=f_\chi\sum_{i=1}^{d_\chi}\pi_i^{(\chi)m}$ and

$$\Phi_\chi(u)=\textstyle\sum_{m=1}^\infty\left(\sum_i\pi_i^{(\chi)m}\right)u^m=\sum_{i=1}^{d_\chi}\pi_i^{(\chi)}u/(1-\pi_i^{(\chi)}u).$$

Therefore we have

$$L(,u\,\chi,\Gamma/\Gamma_0)=\exp\left(\textstyle\sum_i\log(1-\pi_i^{(\chi)}u)\right)/(1-u)^{a\chi}(1-qu)^{a\chi}$$
$$=\textstyle\prod_{i=1}^{d_\chi}(1-\pi_i^{(\chi)}u)/(1-u)^{a\chi}(1-qu)^{a\chi}.$$

Since $Z(u,\Gamma)=\det(E_{2g}-M_l(\pi)u)/(1-u)(1-qu)$, this result gives an algebraic-geometrical explanation of the well-known group-theoretical decomposition of the zeta-function $Z(u,\Gamma)$:

$$Z(u,\Gamma)=Z(u,\Gamma_0)\textstyle\prod_{\chi\neq1}L(u,\chi,\Gamma/\Gamma_0)^{f_\chi}.$$

Correction. In Theorem 2 of the previous paper [1], the functional equations of L-series $L(u,\chi,A/V)$ with $\chi\neq1$ should be corrected as follows:

$$L(1/q^r u,\chi,A/V)=(-1)^{e(\chi)}W(\chi)u^{e(\chi)}L(u,\bar\chi,A/V),$$

where $W(\chi)$ is a constant with $|W(\chi)|=q^{re(\chi)/2}$ and $W(\bar\chi)=\overline{W(\chi)}$.

References

[1] M. Ishida: On zeta-functions and L-series of algebraic varieties, Proc. Japan Acad., **34**, 1-5 (1958).
[2] S. Lang: Sur les séries L d'une variété algébrique, Bull. Soc. Math. France, **84**, 385-407 (1956).
[3] J. Schur: Die algebraischen Grundlagen der Darstellungstheorie der Gruppen, Zürich (1936).
[4] A. Weil: Sur les Courbes Algébriques et les Variétés Qui s'én Déduisent, Paris (1948).
[5] ——: Variétés Abéliennes et Courbes Algébriques, Paris (1948).

3) Here d_χ may be 0. Then the matrix on the right side means 0-matrix.