133. *Multiplication of* (ER)*-Integrable Functions*

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In this note we propose a condition under which multiplication of (ER)-integrable functions is also (ER)-integrable[1] and its applications.

Let R be a set and μ a measure defined on a σ-ring which consists of subsets of R.

Lemma. Let $f(x)$ be an (ER)-integrable function over a μ-measurable set E. And let $g(x)$ be an essentially bounded function defined on E. If there exists a defining fundamental sequence $\{f_n, F_n\}$ of f such that

$$\sum_{n=0}^{\infty} \operatorname*{ess.\,sup.}_{x\in E-F_n} |g(x)| < +\infty,$$

then $f(x)g(x)$ is integrable over E in the sense of Radon-Stieltjes.

In the sequel we assume R is the r-dimensional Euclidian space $(1\leq r<\infty)$ and μ the Lebesgue measure on R. An (ER)-integrable function f over a measurable set E is called of type $(*)$ if there exists a defining fundamental sequence $\{f_n, F_n\}$ of f such that $E-F_n$ is a regular sequence.[2]

Theorem 1. *If $f(x)$ is an (ER)-integrable function of type $(*)$ and $g(x)$ satisfies the Lipschitz condition of order α for some $\alpha, \alpha>0$, on $\overline{E}$, then $f(x)g(x)$ is (ER)-integrable over E.*

If $f(x)$ is an (ER)-integrable function of type $(*)$ over a closed interval $[a, b]$ where $-\infty<a<b<+\infty$, then $f(x)$ is (ER)-integrable over $[a, x]$ if $x \notin \bigcap_{n=0}^{\infty} (\overline{[a, b]-F_n})$.

Theorem 2 (*Integration by parts*).[3] *If $f(x)$ is an (ER)-integrable function of type $(*)$ over $[a, b]$ and $g(x)$ satisfies the Lipschitz condition on $[a, b]$, then we have*

$$(ER)\int_a^b f(x)g(x)\,dx = F(b)g(b) - (ER)\int_a^b F(x)g'(x)\,dx,$$

where $F(x) = (ER)\int_a^x f(x)\,dx.$

1) See K. Kunugi: Application de la méthode des espaces rangés à la théorie de l'intégration. I, Proc. Japan Acad., **32**, 215–220 (1956); and H. Okano: (ER)-integral of Radon-Stieltjes type, Proc. Japan Acad., **34**, 580–584 (1958).

2) See, for example, S. Saks: Theory of the Integral, 106 (1937).

3) S. Nakanishi has obtained an analogous theorem under another assumption. S. Nakanishi: L'intégral (*E. R.*) et la théorie des distributions, Proc. Japan Acad., **34**, 565–570 (1958).

If $f(x)$ is an (ER)-integrable function of type $(*)$ over $[-\pi, +\pi]$, then, from Theorem 1, $f(x)\cos nx$ and $f(x)\sin nx$ are also (ER)-integrable over $[-\pi, +\pi]$. Hence we shall consider the Fourier series of $f(x)$

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx),$$

where

$$a_n=\frac{1}{\pi}(ER)\int_{-\pi}^{+\pi}f(x)\cos nx\,dx\ (n=0, 1, 2, \cdots)$$

and

$$b_n=\frac{1}{\pi}(ER)\int_{-\pi}^{+\pi}f(x)\sin nx\,dx\ (n=1, 2, \cdots).$$

Let $f(x)$ be a function such that $f(0)=0$ and $f(x)=\frac{g(|x|)}{x}$ if $x\neq 0$, where $g(x)$ is a monotone increasing function defined on $0<x\leq\pi$. If $f(x)$ is (ER)-integrable, then the Fourier series of $f(x)$ converges to $\frac{f(x-0)+f(x+0)}{2}$ for every $x\neq 0$. While, if $f(x)$ is not (ER)-integrable, then we have $\lim_{n\to\infty} b_n\neq 0$.