132. (ER)-Integral of Radon-Stieltjes Type

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By the method of ranked spaces Prof. K. Kunugi constructed a new integral[1] which we call *(ER)-integral.* The aim of this note is to study the theory of (ER)-integral of Radon-Stieltjes type on abstract measure spaces.

Let R be a set, $\mathfrak{B}$ a σ-ring which consists of subsets of R and μ a measure defined on $\mathfrak{B}$. A step function

$$f(x)=\sum_{i=1}^{n}\alpha_i\chi_{E_i}(x)\ ^{2)}$$

defined on a μ-measurable set E is called integrable if $\mu(E_i)<+\infty$ for every index i for which $\alpha_i\neq 0$ and the integral of $f(x)$ is defined by

$$\int_E f(x)\,d\mu=\sum_{i=1}^{n}\alpha_i\,\mu(E_i).\ ^{3)}$$

Let $\mathfrak{S}_E$ be the family of all integrable step functions defined on a μ-measurable set E.

It is well known that we obtain integrable functions in the sense of Radon-Stieltjes by completion of the space $\mathfrak{S}_E$ normed by

$$||f||=\int_E |f(x)|\,d\mu.\ ^{4)}$$

Now, according to the method of Prof. K. Kunugi we shall introduce a rank to $\mathfrak{S}_E$ and get the definition of (ER)-integral of Radon-Stieltjes type by completion of the *ranked space* $\mathfrak{S}_E$.[5]

1. *Definitions and some properties.* Let E be a μ-measurable set. For a non-negative integer n, a μ-measurable set F contained in E and an integrable step function f defined on E, $V(F, n; f)$ denotes the family of all integrable step functions $g(x)$ such that there exist two integrable step functions $p(x)$ and $r(x)$ which satisfy the following conditions:

1) K. Kunugi: Application de la méthode des espaces rangés à la théorie de l'intégration. I, Proc. Japan Acad., **32**, 215–220 (1956). T. Ikegami gave a definition of (ER)-integral relative to Haar measure. T. Ikegami: A note on the integration by the method of ranked spaces, Proc. Japan Acad., **34**, 16–21 (1958).

2) $\chi_{E_i}(x)$ denotes the characteristic function of a μ-measurable set E_i and α_i is a real number.

3) We set $0\cdot\infty=0$.

4) See, for example, P. R. Halmos: Measure Theory, Chapter V, New York (1950).

5) See K. Kunugi: *Loc. cit.* and K. Kunugi: Sur les espaces complets et régulièrement complets. I, II, Proc. Japan Acad., **30**, 553–556, 912–916 (1954).

(1) $$g(x)=f(x)+p(x)+r(x).$$

(2) $$r(x)=0 \quad \text{if} \quad x\in F.$$

(3) We have $$\int_E |p(x)|\,d\mu<2^{-n}.$$

(4) We have $$\left|\int_E r(x)\,d\mu\right|<2^{-n}.$$

Let the class of neighbourhoods of f of rank n be the totality of $V(F, n; f)$ such that

$$\mu(E-F)<2^{-n},$$

where F is a μ-measurable set contained in E.

Then $\mathfrak{S}_E$ is a ranked space.

A fundamental sequence in the ranked space $\mathfrak{S}_E$ can be regarded as a sequence $\{f_n, F_n; n=0, 1, 2, \cdots\}$ of pairs of integrable step functions $f_n(x)$ defined on E and μ-measurable sets F_n contained in E which satisfy the following conditions (1) and (2):

(1) i) $$F_n\subseteq F_{n+1}, \quad \text{for any} \quad n.$$

ii) $$\lim_{n\to\infty}\mu(E-F_n)=0.$$

(2) There exist two sequences $p_n(x)$ and $r_n(x)$ of integrable step functions defined on E such that

i) $$f_{n+1}(x)=f_n(x)+p_n(x)+r_n(x),$$

ii) $$r_n(x)=0 \quad \text{if} \quad x\in F_n,$$

iii) $$\sum_{n=0}^{\infty}\int_E |p_n(x)|\,d\mu<+\infty$$

and

iv) $$\sum_{n=0}^{\infty}\left|\int_E r_n(x)\,d\mu\right|<+\infty.$$

Let $u=\{f_n, F_n\}$ be a fundamental sequence. Then $f_n(x)$ tends almost everywhere to a function $f(x)$ on E. By identifying two functions different on a null-set the function $f=f(x)$ is uniquely determined. We denote it by $J(u)$. While, $\int_E f_n(x)\,d\mu$ is a Cauchy sequence of real numbers and, therefore, tends to a real number which is denoted by $I(u)$.

For any n, $J(u)$ is integrable on F_n in the sense of Radon-Stieltjes.

We shall consider the following property (P*) for a fundamental sequence $u=\{f_n, F_n\}$:

(P*) (1) There exists a function $\phi(n)$ of $n(n=0, 1, 2, \cdots)$ such that

i) $$\phi(n)>0,$$

ii) $$\lim_{n\to\infty}\phi(n)=0$$

and

iii) if $\mu(A)\leq m\mu(E-F_n)$ for some positive integer m, then we have

$$\int_A |f_n(x)|\,d\mu \leq m\phi(n).$$

(2) There exists a real number k (independent of n) such that

$$k\mu(E-F_{n+1}) \geq \mu(E-F_n)$$

for any n.

If u_1 and u_2 are two fundamental sequences with (P*), then $J(u_1)=J(u_2)$ implies $I(u_1)=I(u_2)$. Hence, if, for a real valued function $f=f(x)$, there exists a fundamental sequence u with (P*) such that $J(u)=f$, then $f(x)$ is called *(ER)-integrable over E (with respect to μ)* and the integral of $f(x)$ over E, in symbol $(ER)\int_E f(x)\,d\mu$, is defined by

$$(ER)\int_E f(x)\,d\mu = I(u).$$

u is called a defining fundamental sequence of f.

If μ is the 1-demensional Lebesgue measure, then the integral just defined coincides with the one defined by Prof. K. Kunugi.

Our integral has the following properties.

Theorem 1. *We have*

$$(ER)\int_E f(x)\,d\mu = \lim_{n\to\infty}(R)\int_{F_n} f(x)\,d\mu.$$ [6]

Theorem 2. *Let $u=\{f_n, F_n\}$ be a defining fundamental sequence of $f(x)$. And let $\mathfrak{B}_{F_n}$ be the σ-ring which consists of every μ-measurable set contained in F_n and $\mathfrak{A}$ the ring generated by $\bigcup_{n=0}^{\infty}\mathfrak{B}_{F_n}$ and $\{E\}$.*[7] *Then $f(x)$ is (ER)-integrable over every set which belongs to $\mathfrak{A}$.*

Theorem 3. *Let $f(x)$ be (ER)-integrable over E and F.*

(1) *If $E\cap F=0$, then $f(x)$ is (ER)-integrable over $E\cup F$ and we have*

$$(ER)\int_{E\cup F} f(x)\,d\mu = (ER)\int_E f(x)\,d\mu + (ER)\int_F f(x)\,d\mu.$$

(2) *If $E\subseteq F$, then $f(x)$ is (ER)-integrable over $F-E$ and we have*

$$(ER)\int_{F-E} f(x)\,d\mu = (ER)\int_F f(x)\,d\mu - (ER)\int_E f(x)\,d\mu.$$

Theorem 4. *Let $f(x)$ and $g(x)$ be (ER)-integrable functions over E and α and β real numbers. Then $\alpha f(x)+\beta g(x)$ is also (ER)-integrable over E and we have*

$$(ER)\int_E (\alpha f(x)+\beta g(x))\,d\mu = \alpha(ER)\int_E f(x)\,d\mu + \beta(ER)\int_E g(x)\,d\mu.$$

2. *Integrable functions in the sense of Radon-Stieltjes.* Rela-

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6) $(R)\int_{F_n} f(x)\,d\mu$ denotes the Radon-Stieltjes integral of $f(x)$ over F_n.

7) $\{E\}$ denotes the set which consists of only one element E.

tions of Radon-Stieltjes integral and ours are showed in the following theorems.

Theorem 5. *If $f(x)$ is integrable over E in the sense of Radon-Stieltjes, then $f(x)$ is (ER)-integrable over E and we have*

$$(R)\int_E f(x)\,d\mu=(ER)\int_E f(x)\,d\mu.$$

Theorem 6. *The following conditions are equivalent.*

(1) *$f(x)$ is integrable over E in the sense of Radon-Stieltjes.*

(2) *$|f(x)|$ is (ER)-integrable over E.*

(3) *$f(x)$ is (ER)-integrable over every μ-measurable set contained in E.*

3. *A theorem of Radon-Nikodym type.* In this section we assume that $R\in\mathfrak{B}$ and $\mu(R)<+\infty$. It is well known that an absolutely continuous σ-additive finite set-function defined on $\mathfrak{B}$ is represented by indefinite Radon-Stieltjes integrals. Now we propose a condition such that a set-function can be represented by our integral.

Theorem 7. *Let F_n, $n=0, 1, 2, \cdots$, be a sequence of monotone increasing μ-measurable sets such that*

$$\lim_{n\to\infty}\mu(F_n)=\mu(R)$$

and

$$k\mu(R-F_{n+1})\geq\mu(R-F_n)$$

for some k independent of n. And let ν be a finitely additive finite set-function defined on the ring $\mathfrak{A}$ generated by $\bigcup_{n=0}^{\infty}\mathfrak{B}_{F_n}$ and $\{R\}$. If ν satisfies the following conditions:

(1) $$\nu(R)=\lim_{n\to\infty}\nu(F_n).$$

(2) *For any n, ν is σ-additive on $\mathfrak{B}_{F_n}$.*

(3) $$\sum_{n=0}^{\infty}|\nu(F_{n+1}-F_n)|<+\infty.$$

(4) *There exists a positive function $\phi(n)$ of n $(n=0, 1, 2, \cdots)$ such that*

i) $$\lim_{n\to\infty}\phi(n)=0$$

and

ii) *if $A\subseteq F_n$ and $\mu(A)\leq m\mu(R-F_n)$ for some positive integer m, then we have $|\nu|(A)<m\phi(n)$;*[8] *then there exists a function $f(x)$ which is (ER)-integrable over every set which belongs to $\mathfrak{A}$ and we have*

$$\nu(A)=(ER)\int_A f(x)\,d\mu$$

for every $A\in\mathfrak{A}$.

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8) $|\nu|$ denotes the total variation of ν.

If ν is an absolutely continuous σ-additive finite set-function defined on $\mathfrak{B}$, then, by setting $F_n=R$, we have $\mathfrak{A}=\mathfrak{B}$ and, therefore, $f(x)$ is integrable on R in the sense of Radon-Stieltjes from Theorem 6. Hence the above theorem is an extension of the Radon-Nikodym's theorem from Theorem 5.