156. A Generalization of Vainberg's Theorem. II

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3. Semi-ordered linear spaces R and R' are said to be *similar to each other* if there exists a one-to-one correspondence $\varphi: R\ni a\to\varphi(a)\in R'$ between R and R' such that

(3.1) $\qquad \varphi(-a)=-\varphi(a)$ for all $a\in R$;

(3.2) $\qquad \varphi(a)\geq\varphi(b)$ if and only if $a\geq b$.

The correspondence φ fulfilling the conditions (3.1), (3.2) is called a similar correspondence.

A convex set C in R is said to be an *l-vicinity* if

(3.3) *for any* $a\in R$, *there exists a positive number* α *such that* $\alpha a\in C$;

(3.4) $\qquad a\in C,\ |b|\leq|a|$ *implies* $b\in C$;

(3.5) $\qquad a, b\in C,\ |a|\frown|b|=0$ *implies* $a+b\in C$.

If C is a convex l-vicinity then we have $0\in C$ and $a\asymp b\in C$ for any $a, b\in C$.

Now we say that semi-ordered linear spaces R and R' are *almost similar to each other*, if there exist convex l-vicinities $C\subset R$, $C'\subset R'$ and a similar correspondence ψ from C onto C'. For such ψ we have obviously for $a, b\in C$

$$\psi(a\asymp b)=\psi(a)\asymp\psi(b),\quad \psi(|a|)=|\psi(a)|.$$

When R and R' are almost similar to each other, then for any normal manifold N in R (projection operator $[N]$ on R) there exists a normal manifold in R' (resp. a projection operator $[N]'$ on R') such that

$$x\in[N]C \quad\text{if and only if}\quad \psi(x)\in[N]'C'.$$

Therefore we can conclude that *the proper space* $\mathcal{E}$[1] *of* R *is homeomorphic to that* $\mathcal{E}'$ *of* R', *if* R *and* R' *are almost similar to each other*. The converse of this fact, however, is not true in general. But as for modulared semi-ordered linear spaces we can show *the converse of the above holds valid in sufficiently general cases.* This gives appropriateness for our standpoint of discussing the theme of this paper in modulared semi-ordered linear spaces. The proof of the following theorem owes essentially its idea to that of Theorem 62.1 in [1].

1) In fact, $[N]'$ is a projection operator on R' defined by the least normal manifold including all $\psi(x)$ $(x\in[N]C)$.

Theorem 2. If R and R' are monotone complete, simple,[2] *almost finite modulared spaces and their proper spaces are homeomorphic, then they are almost similar to each other.*

Proof. We denote by $C_m(C'_{m'})$ the set of all elements $x \in R$ (resp. $x' \in R'$) such that $m(x) < +\infty$ (resp. $m'(x') < +\infty$). As C_m and $C'_{m'}$ are obviously convex l-vicinities, we need only to show that there exists a one-to-one similar correspondence between C_m and $C'_{m'}$. We denote by $\mathfrak{p}'$ the image of $\mathfrak{p}$ under the homeomorphism between the proper space of R and that of R', and denote also by $[p]'$ (or $[N]'$) the corresponding projection operator on R' to $[p]$ (resp. $[N]$) with respect to this homeomorphism (i.e. $[p]'$ is a projection operator on R' such that $U'_{[p]'} = \{\mathfrak{p}' : \mathfrak{p} \in U_{[p]}\}$ ([1, §8] or [4]).

We shall first show that *for any $0 \lneqq a \in C_m$ there exists an element $x' \in R'$ such that $0 \lneqq x' \in C'_{m'}$ and $m'([p]'x') \le m([p]a)$ for all $[p]$.*

In fact we can find $y' \in R'$ such that $[a]'R' \ni y' \gneqq 0$ and $m'(y') \le m([y']^{-1}a) < +\infty$.[3] If for every $[q] \le [y']^{-1}$ there exists $0 \neq [q_0] \le [q]$ such that $m'([q_0]'y') > m([q_0]a)$, then we can find a system of projectors[4] $[p_\lambda] \le [y']^{-1} (\lambda \in \Lambda)$ such that $m'([p_\lambda]'y') > m([p_\lambda]a)$, $[p_\lambda][p_\gamma] = 0$ if $\lambda \neq \gamma$ and $\bigcup_{\lambda \in \Lambda} [p_\lambda] = [y']^{-1}$. Then we obtain $m'(y') = \sum_{\lambda \in \Lambda} m'([p_\lambda]'y') > \sum_{\lambda \in \Lambda} m([p_\lambda]a) = m([y']^{-1}a)$, which is a contradiction. Therefore we can find a projector $[p_0]$, such that $0 \lneqq [p_0] \le [y']^{-1}$ and $m'([q]'y') \le m([q]a)$ for all $[q] \le [p_0]$. Putting $x' = [p_0]'y'$ we obtain the element satisfying the above condition. For any $C_m \ni a \ge 0$ we denote by D'_a the totality of all elements $R' \ni x' \ge 0$ such that $m'([p]'x') \le m([p]a)$ for all $[p]$. Since $x' \smile y' = [(x'-y')^+]x' + (1 - [(x'-y')^+])y'$, we have for any $x', y' \in D_a$

$$m'([p]'(x' \smile y')) = m'([p]'[(x'-y')^+]x') + m'([p]'(1 - [(x'-y')^+])y')$$
$$\le m([p][(x'-y')^+]^{-1}a) + m([p](1 - [(x'-y')^+])^{-1}y') = m([p]a)$$

for all $[p]$. Thus $x', y' \in D'_a$ implies $x' \smile y' \in D'_a$.

Since m is monotone complete and $m'(x') \le m(a)$ for $x' \in D'_a$ ($0 \lneqq a \in C_m$), there exists $0 \neq b'_a = \bigcup_{x' \in D'_a} x'$. And we have $b'_{[p]a} = [p]'b'_a$ for all $[p]$.

Next we shall prove that

$$m'([p]'b'_a) = m([p]a) \tag{3.6}$$

for all $[p]$. Let $m'([p_0]'b'_a) < m([p_0]a)$ for some $[p_0]$. In consequence

2) A modular m is said to be simple, if $m(a)=0$ implies $a=0$. When m is finite, the assumption of simpleness can be erased, since putting

$$m_1(x) = m(e + |x|)$$

where $e = \bigcup_{x \in R,\, m(x)=0} x$, we obtain a simple, finite, monotone complete modular on R. On the other hand, when a modular m fails to be finite this assumption is essential one in general.

3) $[y']^{-1} (y' \in R')$ is a projection operator on R such that $([y']^{-1})' = [y]'$ (i.e. "$[y'] \to [y']^{-1}$" is the inverse of "$[p] \to [p]'$").

4) $[p] = [\{p\}]$ is said to be *projector*.

of the fact showed above, we have $0 \neq [p_0]'b'_a \in C'_{m'}$. Since m' is almost finite, we can find also a $[q'_0]$ on R', $[q'_0] \leq [p_0]'$ such that $[q_0]'b'_a \in F'_{m'}$ and $m'([q]'b'_a) < m([q']^{-1}a)$ for all $[q'] \leq [q_0]'$. Then we can find a positive number $\alpha > 1$ and $[q'_1]$ ($[q'_1] \leq [q_0]'$) such that $m'(\alpha[q]'b_a) \leq m([q']^{-1}a)$ for all $[q'] \leq [q'_1]$. Hence we have $x' = \alpha[q'_1]b'_a + (1-[q'_1])\, b'_a \in D'_a$ and $x' \gneqq b'_a$ which is a contradiction. Therefore we obtain $m'([p]'b'_a) = m([p]a)$ for all $0 \lneqq a \in C_m$ and $[p]$.

Putting $\psi(a) = b'_{a^+} - b'_{a^-}$, for any $a \in C_m$ we obtain a mapping from C_m into $C'_{m'}$. Considering the converse of ψ, we can see also that the mapping ψ acts onto $C'_{m'}$. Since m and m' are simple, it is easily seen that ψ is a similar correspondence between C_m and $C'_{m'}$ from the definition. Therefore the proof is completed.

From the above theorem we have immediately

Corollary 1. *If R and R' are monotone complete, finite modulared spaces whose proper spaces are homeomorphic, then they are similar to each other.*

Now it comes into question that *a normed*[5] *semi-ordered linear space which is similar (almost similar) to a modulared space can be also modulared space.* Though we can show that the affirmative answer for this question is given, this problem shall be discussed in another paper for avoiding to come off from the present subject.

4. Here we define

Definition. In case R and R' are almost similar to each other, an operator H from R into R' is said to be *splitable with respect to* ψ if it satisfies

$$H(N]x) = [N]'Hx \tag{4.1}$$

for all $x \in R$ and $N \subset R$ ($[N]'$ is a projection operator on R' corresponding to $[N]$ by ψ).

We have shown in Theorem 1 an inequality for a splitable operator H on R into itself, where R is a non-atomic almost finite modulared space. When R and R' are almost similar to each other by ψ, we have similarly for a splitable operator H on R into R' with respect to ψ.

Theorem 3. *Let R and R' be non-atomic semi-ordered linear space and almost similar to each other by a similar correspondence $\psi(R \subset C \xrightarrow{\phi} C' \subset R)$. Furthermore let R' be a modulared space with a modular m' which is almost finite and monotone complete. Then for any splitable operator H with respect to ψ and positive number α, there exist a positive number $\gamma_\alpha > 0$ and an element $0 \leq c_\alpha \in R'$ such that*

5) A modulared space is a normed space at the same time with the norm: $|||x||| = \inf_{m(\xi x) \leq 1} \frac{1}{|\xi|}$ which is equivalent to the modular topologically. But the converse of this is not true in general.

(4.2) $$|H(\alpha x)|\leq c_\alpha+\gamma_\alpha|\psi(x)|$$
for all $x\in C$.

Proof. We put for any positive number α
$$H'(x')=H(\alpha\psi^{-1}(x'))$$
for $x'\in C'$. Since ψ^{-1} is also similar correspondence from C' to C and $\psi^{-1}([M']x')=[M']^{-1}\psi^{-1}(x')$ for any $x'\in C'$ and $M'\subset R'$, H' is also splitable operator on C' into R'. Then by virtue of Remark 2 of Theorem 1 there exist $0\leq c'_\alpha\in R'$ and a positive number γ_α such that
$$H(\alpha\psi^{-1}(x'))=|H'(x')|\leq c'_\alpha+\gamma_\alpha|x'|$$
for all $x'\in C'$. Consequently we have
$$|H(\alpha x)|\leq c'_\alpha+\gamma_\alpha\psi(|x|)$$
for all $x\in C$.

Corollary 2. *If we replace the condition that R and R' are almost similar by "similar to each other by φ" in Theorem* 2, *then we have for any splitable operator from R into R' with respect to φ, there exist a positive number γ and an element $0\leq c'\in R'$ such that*
(4.3) $$|Hx|\leq c+\gamma\varphi(|x|)$$
for all $x\in R$.

Remark 3. For a function $f(u,t)$ introduced in 1, putting $(H(x))(t)=f(x(t),t)-f(0,t)$ we obtain a splitable operator on one function space B consisting of measurable functions on E into another.

Remark 4. In case $\mathfrak{H}\equiv f(u,t)$ operates from L_p to L_{p_1} $(p>0, p_1\geq 1)$, putting $\varphi_p^{p_1}(x(t))=|x(t)|^{\frac{p}{p_1}}\operatorname{sign}|x(t)|$ for any $x(t)\in L_p$, $\varphi_p^{p_1}$ is a similar correspondence from L_p onto L_{p_1}. Then the formula (4.3) corresponding to $\varphi_p^{p_1}$ is nothing but Vainberg's result in [2] introduced in 1. When Young functions Φ and Φ_1 satisfy the Δ_2-condition (i.e. $\Phi(2u)\leq\gamma\Phi(u)$ for all $u\geq u_0\geq 0$, if $\operatorname{mes}(E)<\infty$, and for all $u\geq 0$ if $\operatorname{mes}(E)=+\infty$), $L_\Phi^*(E)$ and $L_{\Phi_1}^*(E)$ are similar to each other by a similar correspondence which coincides with a mapping: $L_\Phi^*(E)\to x(t)\to\Phi_1^{-1}(\Phi(x(t)))\in L_{\Phi_1}^*$ in case $\Phi(u)$ and $\Phi_1(u)$ are positive for any $u>0$. Thus if $\mathfrak{H}\equiv f(u,t)$ operates from $L_\Phi^*(E)$ into $L_{\Phi_1}^*(E)$, we have by Corollary 2
(4.4) $$|f(u,t)|\leq a(t)+r(\Phi_1^{-1}(\Phi(u)))$$
for all $u\in(-\infty,\infty)$ where $a(t)\in L_{\Phi_1}(E)$.[6]

Remark 5. Theorem 2 does not include all cases covered by Vainberg's Theorem, since $L_p(E)$ spaces can not be treated as modulared space, for $0<p<1$. Yet they are similar to a modulared space, for instance, to $L_2(E)$ by correspondence: $L_p(E)\ni x(t)\to|x(t)|^{\frac{p}{2}}\operatorname{sign}\{x(t)\}\in L_2(E)$. Hence Corollary 2 allows to obtain a complete generalization of Vainberg's Theorem by replacing the condition "R is a modulared

6) This result was obtained by Vainberg and Shragin in [3]. In [3] the details for this operator: $\mathfrak{H}\equiv f(u,t)$ were discussed.

space" by the weaker one "R is similar to a modulared space". In this case we have in place of (4.3)

(4.5) $$|Hx| \leq c+\gamma\varphi(\rho(x)) \qquad (x \in R)$$

where ρ is a similar correspondence from R onto itself.

Lastly we consider the most important case, where H operates from R into the conjugate space $\overline{R}$. For operators defined on R into $\overline{R}$ we can consider the following condition:

(4.6) $$H([N]x)=(Hx)[N]^{7)} \quad \text{for all } x \in R,\ N \subset R.$$

A semi-ordered linear space R is said to be *conjugately similar* [1, § 59] if there exists a similar correspondence: $R \ni x \rightarrow x^{\overline{R}} \in \overline{R}$ between R and $\overline{R}$ such that $x^{\overline{R}}(x)=0$, $x \geq 0$ implies $x=0$. For example, L_p-spaces ($p \geq 1$) are the conjugately similar space with the correspondence such that $L_p \in a(t) \geq 0 \rightarrow a(t)^{\frac{p}{q}} \in L_q$ where $\frac{1}{p}+\frac{1}{q}=1$. When R is a conjugately similar space, the condition (4.6) is equivalent to the fact that H is splitable with respect to the conjugately similar correspondence [1, §§ 22, 60]. Let R be a conjugately similar space. Then by virtue of the fundamental theorem established by H. Nakano [1, § 60] which shows the essential relationships between modulared spaces and conjugately similar spaces, *we can define finite monotone complete modulars on R and on $\overline{R}$.*[8] Therefore we have by virtue of Corollary 2 immediately

Theorem 4. *Let R be non-atomic conjugately similar space with correspondence $R \ni x \rightarrow x^{\overline{R}} \in \overline{R}$. If an operator H on R into $\overline{R}$ satisfies* (4.6), *then for some $0 \leq \overline{c} \in \overline{R}$ and $\gamma > 0$ we have*

(4.7) $$|Hx| \leq \overline{c}+\gamma|x|^{\overline{R}} \qquad \textit{for all } x \in R.$$

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7) For $\overline{a} \in \overline{R}$ and $N \subset R$ we denote by $\overline{a}[N]$ a functional such that $(\overline{a}[N])x = \overline{a}([N]x)$ for all $x \in R$.

8) The converse of this is also valid [1, § 62].