154. Note on Idempotent Semigroups. V. Implications of Two Variables

By Miyuki YAMADA
Shimane University

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§ 1. This note is the continuation of the previous papers (Kimura [1, 3, 4]; Yamada and Kimura [2]). Any terminology without definition should be referred to them.

The purpose of this note is to present the classification of all implications of two variables on idempotent semigroups and some relevant matters.

The proofs of any lemmas and theorems are all omitted, which will be given in detail elsewhere.[1]

Let $X=\{x_1, x_2,\cdots,x_n\}$ be a set where each element x_i is called a variable. Let S be an idempotent semigroup. Then, we call every element $x_{i1}x_{i2}\cdots x_{im}$ of the free semigroup generated by X a *polynomial* of n variables $x_1,x_2,\cdots,x_n$ on S when we regard each element of X as a variable on S and $x_{i1}x_{i2}\cdots x_{im}$ as the product of $x_{i1}x_{i2}\cdots x_{im}$ with respect to the multiplication in S, and denote by $f(x_1, x_2,\cdots, x_n)$, etc.

Take up four families of polynomials, $\{f_i: i\in I\}$, $\{g_i: i\in I\}$, $\{f_j^*: j\in J\}$ and $\{g_j^*: j\in J\}$, of the same variables $x_1, x_2,\cdots, x_n$, and consider a relation defined by

(P.I) $\{f_i(x_1, x_2,\cdots, x_n)=g_i(x_1, x_2,\cdots, x_n) : i\in I\}$ implies $\{f_j^*(x_1, x_2,\cdots, x_n)=g_j^*(x_1, x_2,\cdots, x_n) : j\in J\}$.

Such a relation is called a *polynomial implication*, or more simply an *implication*, of n variables $x_1, x_2,\cdots, x_n$ on idempotent semigroups, and denoted by

$$\{f_i(x_1, x_2,\cdots, x_n)=g_i(x_1, x_2,\cdots, x_n) : i\in I\} \Rightarrow$$
$$\{f_j^*(x_1, x_2,\cdots, x_n)=g_j^*(x_1, x_2,\cdots, x_n) : j\in J\}.$$

For example, $\{xy=yx, xyx=yx\}\Rightarrow x=y$ is an implication of two variables x, y. Particularly, an implication is called *"trivial"* if it's satisfied by all idempotent semigroups. Further, if $\{f_i(x_1, x_2,\cdots, x_n) =g_i(x_1, x_2,\cdots, x_n) : i\in I\}$ in (P.I) consists of only trivial identities[2] on idempotent semigroups, then any idempotent semigroup satisfies (P.I) if and only if it satisfies identities $\{f_j^*(x_1, x_2,\cdots, x_n)=g_j^*(x_1, x_2,\cdots, x_n) : j\in J\}$, and therefore in this case (P.I) is called especially a *family of identities*.

1) This is an abstract of the paper which will appear elsewhere.

2) An equality $f(x_1, x_2,\cdots, x_n)=g(x_1, x_2,\cdots, x_n)$ is called a trivial identity on idempotent semigroups if it becomes an identity for all idempotent semigroups.

The present paper deals exclusively with implications, identities and equalities, of two variables x, y on idempotent semigroups, and the modifier "of two variables x, y on idempotent semigroups" will be omitted throughout the paragraphs 2 and 3.

§ 2. As the first step, we shall clarify here the mutual relation between all families of equalities. We shall say two families $\{f_i=g_i : i\in I\}$ and $\{f_j^*=g_j^* : j\in J\}$ of equalities to be equivalent if both $\{f_i=g_i : i\in I\}\Rightarrow\{f_j^*=g_j^* : j\in J\}$ and $\{f_j^*=g_j^* : j\in J\}\Rightarrow\{f_i=g_i : i\in I\}$ are trivial implications.

The following lemma is easily obtained by direct calculations.

Lemma 1. *Every family of equalities is equivalent to one of the following* (0)–(15):

(0) *The family of all trivial identities,* (1) $x=y$, (2) $x=yxy$, (3) $\begin{Bmatrix} x=yx \\ y=xy \end{Bmatrix}$, (4) $\begin{Bmatrix} x=xy \\ y=yx \end{Bmatrix}$, (5) $y=xyx$, (6) $\begin{Bmatrix} x=xyx \\ y=yxy \end{Bmatrix}$, (7) $x=yx$, (8) $x=xy$, (9) $xy=yx$, (10) $y=xy$, (11) $y=yx$, (12) $x=xyx$, (13) $xy=xyx$, (14) $xy=yxy$, (15) $y=yxy$.

We denote by $[i]$ the class of all families, each family of which is equivalent to (i), where $i=0, 1, 2, \cdots, 15$. Now, we introduce a multiplication $\cdot$ and an ordering $\leqq$ into the system consisting of classes $[0]$–$[15]$ as follows:

(M.R) $[i]\cdot[j]=[s]$ if and only if $\{(i), (j)\}$ is equivalent to (s),

(O.R) $[i]\geqq[j]$ if and only if $[i]\cdot[j]=[j]$.

Then this system becomes a semilattice, and consequently we have

Theorem 1. *All families of equalities are classified into* 16 *distinct classes* $[0]$–$[15]$, *each of which consists of equivalent families. Further, by the multiplication defined in* (*M.R*), *the system consisting of such classes becomes a semilattice.*

§ 3. Let ϕ and ψ be two implications. Then, such two implications ϕ and ψ are said to be equivalent, if, whenever the former is satisfied by an idempotent semigroup S, the latter is also satisfied by the same idempotent semigroup S, and if the converse holds.

Lemma 2. *Let* $\{f_{1i}=g_{1i} : i\in I\}$ *and* $\{f_{1j}^*=g_{1j}^* : j\in J\}$ *be two families of equalities, which are equivalent to* $\{f_{2p}=g_{2p} : p\in P\}$ *and* $\{f_{2q}^*=g_{2q}^* : q\in Q\}$ *respectively. Then, the two implications* $\{f_{1i}=g_{1i} : i\in I\}\Rightarrow\{f_{1j}^*=g_{1j}^* : j\in J\}$ *and* $\{f_{2p}=g_{2p} : p\in P\}\Rightarrow\{f_{2q}^*=g_{2q}^* : q\in Q\}$ *are equivalent to each other.*

Combining Lemmas 1 and 2, we have

Theorem 2. *Every implication is equivalent to one of the implications* $\{(i)\Rightarrow(j) : i, j=0, 1, 2, \cdots, 15\}$.

Now, pick up any element (i^*) from each class $[i]$ in Theorem 1.

Then, there exist the following relations between the families $\{(i^*) : i=0, 1, 2, \cdots, 15\}$ and the implications $\{(i^*)\Rightarrow(j^*) : i, j=0, 1, 2, \cdots, 15\}$.

Lemma 3. *$(0^*)\Rightarrow(i^*)$, where $i=0,1,2,\cdots$, or 15, is a family of identities.*

Lemma 4. *If $[i]\leqq[j]$, then $(i^*)\Rightarrow(j^*)$ is a trivial implication.*

Lemma 5. *If $[i]\leqq[j]$ and $[s]\leqq[k]$, then $(j^*)\Rightarrow(s^*)$ implies $(i^*)\Rightarrow(k^*)$.*

Lemma 6. *If $[i]\cdot[j]=[s]$, then $(i^*)\Rightarrow(s^*)$ is equivalent to $(i^*)\Rightarrow(j^*)$.*

An idempotent semigroup is called (I) *trivial*, (II) *right* (*left*) *singular*, (III) *rectangular*, (IV) *commutative*, (V) *right* (*left*) *regular*, respectively, if it satisfies the following corresponding implications: (I) $xy=xy\Rightarrow x=y$, (II) $xy=xy\Rightarrow xy=y$ ($xy=xy\Rightarrow xy=x$), (III) $xy=xy\Rightarrow xyx=x$, (IV) $xy=xy\Rightarrow xy=yx$, (V) $xy=xy\Rightarrow xyx=yx$ ($xy=xy\Rightarrow xyx=xy$).[3)]

Under these definitions, using Lemmas 2–6 and Theorem 2, we obtain the following main theorem.

Theorem 3. *Every implication is equivalent to* (I) *triviality* (*t*), (II) *right singularity* (*r.s*), (III) *left singularity* (*l.s*), (IV) *rectangularity* (*r.a*), (V) *commutativity* (*c*), (VI) *right regularity* (*r.r*), (VII) *left regularity* (*l.r*) *or* (VIII) *a trivial implication* (*t.i*).

In fact, this result is stated more precisely by the following diagram.

$\Rightarrow$	1*	2*	3*	4*	5*	6*	7*	8*	9*	10*	11*	12*	13*	14*	15*	0*
1*	*t.i*	*t.i*	*t.i*	*t.i*	*t.i*	*t.i*	*t.i*	*t.i*	*t.i*	*t.i*	*t.i*	*t.i*	*t.i*	*t.i*	*t.i*	*t.i*
2*	*r.a*	*t.i*	*r.a*	*r.a*	*r.a*	*r.a*	*t.i*	*t.i*	*t.i*	*r.a*	*r.a*	*t.i*	*t.i*	*t.i*	*r.a*	*t.i*
3*	*l.r*	*l.r*	*t.i*	*l.r*	*l.r*	*t.i*	*t.i*	*l.r*	*l.r*	*t.i*	*l.r*	*t.i*	*l.r*	*t.i*	*t.i*	*t.i*
4*	*r.r*	*r.r*	*r.r*	*t.i*	*r.r*	*t.i*	*r.r*	*t.i*	*r.r*	*r.r*	*t.i*	*t.i*	*t.i*	*r.r*	*t.i*	*t.i*
5*	*r.a*	*r.a*	*r.a*	*r.a*	*t.i*	*r.a*	*r.a*	*r.a*	*t.i*	*t.i*	*t.i*	*r.a*	*t.i*	*t.i*	*t.i*	*t.i*
6*	*c*	*c*	*r.r*	*l.r*	*c*	*t.i*	*r.r*	*l.r*	*c*	*r.r*	*l.r*	*t.i*	*l.r*	*r.r*	*t.i*	*t.i*
7*	*l.s*	*l.r*	*r.a*	*l.s*	*l.s*	*r.a*	*t.i*	*l.r*	*l.r*	*r.a*	*l.s*	*t.i*	*l.r*	*t.i*	*r.a*	*t.i*
8*	*r.s*	*r.r*	*r.s*	*r.a*	*r.s*	*r.a*	*r.r*	*t.i*	*r.r*	*r.s*	*r.a*	*t.i*	*t.i*	*r.r*	*r.a*	*t.i*
9*	*r.a*	*r.a*	*r.a*	*r.a*	*r.a*	*r.a*	*r.a*	*r.a*	*t.i*	*r.a*	*r.a*	*r.a*	*t.i*	*t.i*	*r.a*	*t.i*
10*	*l.s*	*l.s*	*r.a*	*l.s*	*l.r*	*r.a*	*r.a*	*l.s*	*l.r*	*t.i*	*l.r*	*r.a*	*l.r*	*t.i*	*t.i*	*t.i*
11*	*r.s*	*r.s*	*r.s*	*r.a*	*r.r*	*r.a*	*r.s*	*r.a*	*r.r*	*r.r*	*t.i*	*r.a*	*t.i*	*r.r*	*t.i*	*t.i*
12*	*t*	*c*	*r.s*	*l.s*	*t*	*r.a*	*r.r*	*l.r*	*c*	*r.s*	*l.s*	*t.i*	*l.r*	*r.r*	*r.a*	*t.i*
13*	*r.s*	*r.s*	*r.s*	*r.a*	*r.s*	*r.a*	*r.s*	*r.a*	*r.r*	*r.s*	*r.a*	*r.a*	*t.i*	*r.r*	*r.a*	*t.i*
14*	*l.s*	*l.s*	*r.a*	*l.s*	*l.s*	*r.a*	*r.a*	*l.s*	*l.r*	*r.a*	*l.s*	*r.a*	*l.r*	*t.i*	*r.a*	*t.i*
15*	*t*	*t*	*r.s*	*l.s*	*c*	*r.a*	*r.s*	*l.s*	*c*	*r.r*	*l.r*	*r.a*	*l.r*	*r.r*	*t.i*	*t.i*
0*	*t*	*t*	*r.s*	*l.s*	*t*	*r.a*	*r.s*	*l.s*	*c*	*r.s*	*l.s*	*r.a*	*l.r*	*r.r*	*r.a*	*t.i*

3) Since $xy=xy$ is a trival identity, these implications are essentially the same as identities. For the structure of idempotent semigroups satisfying the implication (I), (II), (III), (IV) or (V), see Kimura [1] and McLean [5].

References

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