153. *Homomorphisms of a Left Simple Semigroup onto a Group*

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(Comm. by K. SHODA, M.J.A., Dec. 12, 1958)

Cohn, in his paper [1], defined d-semigroups as semigroups S satisfying the following conditions:
(1) if $a, b \in S$, then $a=xb$ for some $x \in S$,
(2) if $a, b \in S$, then either $a=b$ or $a=by$ or $b=ay$ for some $y \in S$,
(3) S contains no idempotent,
and then he characterized the kernels of homomorphisms of a d-semigroup onto a group.

In this note, we show that a similar result holds for left simple semigroups, that is, semigroups satisfying the condition (1) only.

In this note, S *denotes always a left simple semigroup.*

A subsemigroup T of S is said to be *left unitary* in S, if T contains, with any a, b, all solutions x in S of the equation $ax=b$. (This definition is due to Dubreil [2]. Cohn uses the word 'closed' in the sense of 'right and left unitary'.) Also, a subsemigroup T of S is said to be *normal* in S, if $xT \subseteq Tx$ for any $x \in S$.

In S, we define a set U by

$$U=\{x \in S;\ xa=a \text{ for some } a \in S\}.$$

U is non-empty, since S satisfies the condition (1). Also, we define a set V by

$$V=\{x \in S;\ ux=u' \text{ for some } u, u' \in U\}.$$

Lemma 1. $U \subseteq V$.

Proof. If $u \in U$, there exists an element $a \in S$ such that $ua=a$. Then we have also $u^2a=ua=a$, and so $u^2 \in U$. But u is a solution of the equation $ux=u^2$ and so we have $u \in V$.

By Lemma 1, V is also non-empty.

Now we consider the subsemigroup I generated by the set V, and call it the *core* of S. Thus every element of the core I can be represented by a finite product of elements in V.

Lemma 2. *Given $x \in S$ and $v \in V$, there exists an element $v' \in V$ such that $xv=v'x$.*

Proof. By the condition (1), there exists an element $v' \in S$ such that $xv=v'x$. Since $v \in V$, there exist two elements $u_1, u_2 \in U$ such that $u_1v=u_2$. Then, since $u_1, u_2 \in U$, there exist elements $a, b \in S$ such that $u_1a=a$ and $u_2b=b$. Using the condition (1) again, we can consider an element $s \in S$ such that $x=su_1$, and then an element $p \in S$ such that

$su_2=ps$. Furthermore, we can consider an element $q\in S$ such that $qv'=p$. Then we have

$$p(sb)=su_2b=sb,$$
$$q(v'sa)=psa=su_2a=su_1va=xva=v'xa=v'su_1a=v'sa.$$

Therefore we have $p, q\in U$. Hence, by $qv'=p$, we have $v'\in V$.

Lemma 3. *I is a normal subsemigroup.*

Proof. Suppose that $i\in I$ and $x\in S$. Then i can be represented by the form

$$i=v_1v_2\cdots v_n \qquad v_1, v_2,\cdots, v_n\in V.$$

By Lemma 2, there exist n elements $w_1, w_2,\cdots, w_n\in V$ such that $xv_1=w_1x$, $xv_2=w_2x,\cdots$, $xv_n=w_nx$. Then we have

$$xi=xv_1v_2\cdots v_n=w_1xv_2\cdots v_n=w_1w_2x\cdots v_n=\cdots$$
$$=w_1w_2\cdots w_nx.$$

Therefore, since $w_1w_2\cdots w_n\in I$, we have $xi\in Ix$. Hence we have $xI\subseteq Ix$ for any $x\in S$.

Lemma 4. *If $v\in V$, $i\in I$ and $vx=i$, then we have $x\in I$.*

Proof. Since $v\in V$, there exist two elements $u_1, u_2\in U$ such that $u_1v=u_2$. And then, by the condition (1), there exists an element $p\in S$ such that $p(u_2x)=x$. Since $(pu_2)x=x$, we have $pu_2\in U$, and so $p\in V$. Now we have

$$x=pu_2x=pu_1vx=pu_1i.$$

Therefore, x, with $i\in I$, can be represented by a finite product of elements in V, since $p\in V$ and $u_1\in U\subseteq V$. Hence we have $x\in I$.

Lemma 5. *I is a left unitary subsemigroup of S.*

Proof. Suppose that $i\in I$, $j\in I$ and $ix=j$. Then i can be represented by the form

$$i=v_1v_2\cdots v_n \qquad v_1, v_2,\cdots, v_n\in V.$$

We show $x\in I$ by induction with respect to n, the number of terms of the above expression. If $n=1$, then i itself is an element of V, and so, by Lemma 4, we have $x\in I$. Now we consider the general case. We set

$$v_2\cdots v_nx = x'.$$

Then we have

$$v_1x' =v_1v_2\cdots v_nx=ix=j.$$

Therefore, by Lemma 4, we have $v_2\cdots v_nx=x'\in I$, and hence, by the induction hypothesis, we have $x\in I$.

By Lemmas 3 and 5 we obtain the following

Theorem 1. *The core I is a normal and left unitary subsemigroup of S.*

Now we consider a homomorphism of a left simple semigroup S onto a group G. Then we have the following

Theorem 2. *Let θ be a homomorphism of S onto a group G with unit-element* 1. *Then the kernel $\theta^{-1}(1)$ is a normal and left unitary* *subsemigroup of S containing the core I.*

Proof. If $x, y \in \theta^{-1}(1)$, then $\theta(xy)=\theta(x)\theta(y)=1$, and so $xy \in \theta^{-1}(1)$. Hence $\theta^{-1}(1)$ is a subsemigroup. If $x, y \in \theta^{-1}(1)$ and $xz=y$, then we have $1=\theta(y)=\theta(xz)=\theta(x)\theta(z)=\theta(z)$, and so we have $z \in \theta^{-1}(1)$. Hence $\theta^{-1}(1)$ is left unitary. Let t be any element of S. For $x \in \theta^{-1}(1)$, there exists an element $s \in S$ such that $tx=st$. Then we have $\theta(t)=\theta(t)\theta(x)=\theta(tx)=\theta(st)=\theta(s)\theta(t)$ and so $\theta(s)=1$, that is, $s \in \theta^{-1}(1)$. Hence $t\theta^{-1}(1) \subseteq \theta^{-1}(1)t$, and so $\theta^{-1}(1)$ is normal. If $u \in U$, then there exists an element $a \in S$ such that $ua=a$. Then we have $\theta(a)=\theta(ua)=\theta(u)\theta(a)$ and so $\theta(u)=1$. Hence U is contained in $\theta^{-1}(1)$. If $v \in V$, then there exist two elements $u_1, u_2 \in U$ such that $u_1v=u_2$. But u_1 and u_2, being elements of U, are elements of $\theta^{-1}(1)$, and so we have $v \in \theta^{-1}(1)$ since $\theta^{-1}(1)$ is left unitary. Therefore $\theta^{-1}(1)$ is a subsemigroup which contains V, and hence $\theta^{-1}(1)$ contains the core I.

Conversely, let us consider a normal and left unitary subsemigroup N containing the core I.

Lemma 6. *N is right unitary, that is, $a, b \in N$ and $xa=b$ imply $x \in N$.*

Proof. By the normality of N, we have $xa \in xN \subseteq Nx$. Therefore there exists an element $c \in N$ such that $xa=cx$. And so we have $cx=b$. Since N is left unitary, this equality implies $x \in N$.

Lemma 7. *$a \in Na$ for any $a \in S$.*

Proof. By the condition (1) there exists an element $u \in S$ such that $ua=a$. By the definition of the set U, we have $u \in U$. But, since $U \subseteq V \subseteq I \subseteq N$, we have $u \in N$. Hence $a=ua \in Na$.

Lemma 8. *$N=Nx$ if and only if $x \in N$.*

Proof. Suppose that $x \in N$. By the condition (1), for $n \in N$ there exists an element $m \in S$ such that $n=mx$. By Lemma 6, this element m belongs to N. Therefore we have $N \subseteq Nx$. The inverse inclusion $Nx \subseteq N$ holds also, since N is a subsemigroup. Hence we have $N=Nx$. Conversely, suppose that $N=Nx$. Then, for $n \in N$, there exists an element $m \in N$ such that $n=mx$. Then we have $x \in N$, since N is left unitary.

Lemma 9. *If $Na \frown Nb \neq \phi$, then $Na=Nb$.*

Proof. Suppose that $c \in Na \frown Nb$. Then there exist two elements $n_1, n_2 \in N$ such that $c=n_1a=n_2b$. Hence, by Lemma 8, we have $Na=Nn_1a=Nc=Nn_2b=Nb$.

By Lemmas 7 and 9, the different sets Na define a partition of S. The totality of these different sets Na is denoted by $\mathfrak{N}$.

Now, we define a composition $\circ$ in $\mathfrak{N}$ as follows:

$$(Na)\circ(Nb) = Nab.$$

By assumption, N is a normal subsemigroup and so

$$NaNb \subseteq NNab \subseteq Nab.$$

Therefore the product $(Na)\circ(Nb)$ can be regarded as the class containing $NaNb$, and hence this product is uniquely determined irrespective of the choice of the elements a and b. Moreover

$$(Na\circ Nb)\circ Nc = Nabc = Na\circ(Nb\circ Nc)$$

and hence $\mathfrak{N}$ is a semigroup with respect to the composition .

Lemma 10. *The semigroup $\mathfrak{N}$ is a group.*

Proof. By Lemma 8, N is an element of $\mathfrak{N}$. Let Na be an element of $\mathfrak{N}$. Then there exists an element $u\in S$ such that $ua=a$. The element u belongs to U and so belongs to N. Therefore, by Lemma 8, we have $N=Nu$, and so

$$N\circ Na = Nu\circ Na = Nua = Na.$$

Hence N is a left unit-element of the semigroup $\mathfrak{N}$. Moreover, there exists an element $b\in S$ such that $ba=u$. And so

$$Nb\circ Na = Nba = Nu = N.$$

Hence $\mathfrak{N}$ is a group with unit element N (cf. Zassenhaus [3]).

Now we consider a mapping

$$\theta : a \rightarrow Na$$

of S onto $\mathfrak{N}$. θ is evidently a homomorphism.

Lemma 11. *The kernel of the homomorphism θ is N.*

Proof. Let K be the kernel of θ. By definition, $\theta(a)=Na$. Since N is the unit-element of the group $\mathfrak{N}$, $a\in K$ is equivalent to $Na=N$. But, by Lemma 8, the latter equality is equivalent to $a\in N$. Hence we have $K=N$.

By Lemmas 10 and 11, we obtain the following

Theorem 3. *If N is a normal and left unitary subsemigroup of S containing the core I, then there exists a homomorphism θ of S onto a group such that the kernel of θ is N.*

References

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[3] H. Zassenhaus: Lehrbuch der Gruppentheorie, Leibzig (1937).