150. On the Singular Integrals. IV*)

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1. This is a continuation of the previous paper [4, III]. The purpose of this paper is to show the reciprocal formula of the Hilbert operator. The method of proof is a so-called complex variable method which is different from that of the previous one quitely. As an application, we can establish some results for analytic functions in a half-plane.

Let $g(x)$ be a real valued measurable function over $(-\infty, \infty)$ we put

$$C(z, g)=\frac{1}{2\pi i}\int_{-\infty}^{\infty} g(t)\frac{dt}{t-z}, \tag{1.1}$$

$$P(z, g)=\frac{1}{\pi}\int_{-\infty}^{\infty} g(t)\frac{y\,dt}{(t-x)^2+y^2} \tag{1.2}$$

$$\widetilde{P}(z, g)=-\frac{1}{\pi}\int_{-\infty}^{\infty} g(t)\frac{(t-x)dt}{(t-x)+y^2}. \tag{1.3}$$

We shall call $C(z, g)$ and $P(z, g)$ integrals of Cauchy type and Poisson type respectively, associated with the function $g(x)$. We observe also that

$$2C(z, g)=P(z, g)+i\widetilde{P}(z, g). \tag{1.4}$$

We have then

Theorem 1. *Let $g(x)$ belong to L_μ^p $(p\geqq 1,\ 0\leqq\alpha<1)$; then we have*

$$\text{(S)-}\lim_{y\to 0} P(z, g)=g(x), \quad a.e. \tag{1.5}$$

$$\lim_{y\to 0}\int_{-\infty}^{\infty}\frac{|P(z, g)-g(x)|^p}{1+|x|^\alpha}dx=0, \tag{1.6}$$

where the sign (S) *means that the limit exists along a Stoltz' path—as an angular limit.*

Theorem 2. *Let $g(x)$ belong to L_μ^p $(p>1,\ 0\leqq\alpha<1)$ or $g(x)$ and $\widetilde{g}(x)$ both belong to L_μ $(0\leqq\alpha<1)$. Then we have also*

$$\text{(S)-}\lim_{y\to 0} \widetilde{P}(z, g)=\widetilde{g}(x), \quad a.e. \tag{1.7}$$

$$\lim_{y\to 0}\int_{-\infty}^{\infty}\frac{|\widetilde{P}(z, g)-\widetilde{g}(x)|^p}{1+|x|^\alpha}dx=0. \tag{1.8}$$

For this purpose it is enough to prove

Theorem 3. *Under the assumption of Theorem 2 we have*

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*) Here we state the result without proof. The detailed argument will appear in Jour. Fac. Sci. Hokkaidô University.

(1.9) $$P(z, \tilde{g}) = \tilde{P}(z, g).$$

From Theorems 2 and 3 we have

Theorem 4. *Under the assumption of Theorem 2 we have*

(1.10) $$\frac{1}{2\pi i}\int_{-\infty}^{\infty} g(t)\frac{dt}{t-z} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} i\tilde{g}(t)\frac{dt}{t-z}.$$

As an immediate result we have a desired reciprocal formula.

Theorem 5. *Under the assumption of Theorem 2 we have*

(1.11) $$(\tilde{\tilde{g}})(x) = -g(x), \quad a.e.$$

2. Let $f(z)$, $z=x+iy$, be analytic in a half-plane $y>0$. If the limit

(2.1) $$\lim_{y\to 0} f(x+iy) = f(x)$$

exists for almost all x, $f(x)$ will be called the limit function of $f(z)$. If $g(x)=f(x)$ is the limit function of a function analytic for $y>0$, such that

(2.2) $$f(z) = C(z, g), \quad \text{or} \quad f(z) = P(z, g),$$

then we shall say that $f(z)$ is represented by its proper Cauchy or its proper Poisson integral, omitting the adjective "proper" if no confusion arises. By $\mathfrak{H}_\mu^p$ we denote the class of functions $f(z)$ analytic in a half-plane $y>0$ such that the integral

(2.3) $$\|f(x+iy)\|_{p,\mu} = \left(\int_{-\infty}^{\infty} \frac{|f(x+iy)|^p}{1+|x|^\alpha}dx\right) < \text{const.}$$

for $0<y<\infty$.

If we put $\alpha=0$ in (2.2), we obtain the ordinary class $\mathfrak{H}^p$. For this class, there is a study of Paley-Wiener [6] and Hill-Tamarkin [3]. Extension of their result to our class $\mathfrak{H}_\mu^p$ is the purpose of the second half part of this paper. We have

Theorem 6. *Under the assumption of Theorem 2, if we put*

(2.4) $$f(z) = 2C(z, g),$$

then $f(z)$ is analytic in a half-plane $y>0$; its limit function exists as an angular limit and equals to

(2.5) $$f(x) = g(x) + i\tilde{g}(x).$$

Furthermore $f(z)$ is representable by its Cauchy integral.

Theorem 7. *Let $f(z)$ be analytic in a half-plane $y>0$ and have a limit function $f(x)$ which belong to L_μ^p ($p\geqq 1$, $0\leqq\alpha<1$). Then if $f(z)$ is represented by its Cauchy integral, we have*

(2.6) $$(\widetilde{\mathfrak{R}f}) = \mathfrak{I}f \quad \textit{and} \quad (\widetilde{\mathfrak{I}f}) = -\mathfrak{R}f.$$

Theorem 8. *Let $f(z)$ be analytic in a half-plane $y>0$ and have a limit function of L_μ^p ($p>1$, $0\leqq\alpha<1$). Furthermore let this limit function exist as an angular limit on a point of the set with a positive measure. Then $f(z)$ is represented by its Cauchy integral.*

To prove this theorem we need Theorem 5 and the unicity theorem of Lusin-Privaloff [5] and F. and M. Riesz' theorem [7].

Theorem A. *Let $f(z)$ be analytic in the interior of a unit circle. Let $f(z)$ have an angular limit equal to a constant c for a point of the set with a positive measure which is situated on a circumference of this circle. Then $f(z)$ is identically equal to this constant c.*
As for equivalency of the integral representation of the Cauchy type and that of the Poisson type, there is a study of G. Fichtenholtz [1] in a unit circle and that of Hill-Tamarkin [3] in a half-plane for the class $\mathfrak{H}^p$. Now we prove the following:

Theorem 9. *Let $f(z)$ be analytic in a half-plane $y>0$ and have a limit function which belongs to L_μ^p $(p\geqq 1, 0\leqq \alpha<1)$. Then whenever $f(z)$ is represented by its Cauchy integral, it is also represented by its Poisson integral and vice versa.*

REMARK 1. By Theorem 9, $f(z)$ of Theorems 6, 8 and 9 belongs to the class $\mathfrak{H}_\mu^p$.

REMARK 2. In Theorem 8, the case $p=1$ is an open question. If we assume that $\tilde{f}(x)$ also belongs to L_μ, our conclusion is also true, but this additional condition is somewhat strong.

3. In this section we state the result concerning the class $\mathfrak{H}_\mu^p$. This is a converse of the preceding one. The key point is to find a limit function in each functional space. We need two theorems of Paley-Wiener [6] as a base of our arguments.

Theorem B. *Let $f(z)$ belong to $\mathfrak{H}_\mu^p$ $(p=2)$ in an upper half-plane. Then for any given $y_0>0$ we have*

$$f(z+iy_0)=\frac{1}{2\pi i}\int_{-\infty}^{\infty} f(t+iy_0)\frac{dt}{t-z} \tag{3.1}$$

and

$$f(z+iy_0)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(t+iy_0)\frac{y\,dt}{(t-x)^2+y^2} \tag{3.2}$$

for all $y>0$, $z=x+iy$.

Theorem C. *The two following classes of analytic functions are identical:*

(1) *the class of all functions $f(x+iy)$ analytic for $y>0$ such that*

$$\int_{-\infty}^{\infty} |f(x+iy)|^2\,dx<\text{const.}\ [0<y<\infty]; \tag{3.3}$$

(2) *the class of all functions defined by*

$$f(x+iy)=\underset{A\to\infty}{\text{l.i.m.}}\int_{-A}^{0} f(t)\,e^{t(x+iy)}\,dt \tag{3.4}$$

where $f(t)$ belongs to L^2 over $(-\infty,\infty)$.

We begin to establish the following three theorems:

Theorem 10. *Let $f(z)$ belong to $\mathfrak{H}_\mu^p$ $(p\geqq 1,\ 0\leqq\alpha<1)$. Then the conclusion of Theorem B is also true.*

Theorem 11. *Let $f(z)$ belong to $\mathfrak{H}^p_\mu$ $(p\geqq 1,\ 0\leqq\alpha<1)$. Then for any positive number η,*

$$\lim_{z\to\infty} f(z)=o(1),\ \textit{unif. in } y\geqq\eta>0. \tag{3.5}$$

Theorem 12. *Let $f(z)$ belong to $\mathfrak{H}^p_\mu$ $(p\geqq 1,\ 0\leqq\alpha<1)$. Then this $f(z)$ can be written as*

$$f(z)=B_f(z)H(z), \tag{3.6}$$

where $H(z)$ belongs to the same class $\mathfrak{H}^p_\mu$, does not vanish in a half-plane $y>0$, and

$$B_f(z)=\prod_{(\nu)}\frac{z-z_\nu}{z-\bar{z}_\nu}\,\frac{\bar{z}_\nu-i}{z_\nu+i}, \tag{3.7}$$

where $\{z_\nu\}$ is a sequence of zeros of $f(z)$ in $y>0$. The $B_f(z)$ is called a Blaschke product associated with $f(z)$ and has a following properties:

$$|B_f(z)|\leqq 1 \quad \textit{for all} \quad y>0, \tag{3.8}$$

$$\text{(S)-}\lim_{y\to 0} B_f(z)=1, \quad \textit{a.e.x.} \tag{3.9}$$

This is a special case of the theorem of R. M. Gabriel [2], and if we put $\alpha=0$, we obtain the result of Hill-Tamarkin.

Then if we put with a given $f(z)$ of $\mathfrak{H}^p_\mu$

$$F(z)=f(z)/(z+i)^2 \tag{3.10}$$

$F(z)$ belongs to the same class $\mathfrak{H}^p$ because $(z+i)^2$ is an analytic function in an upper half-plane and has no zero point there. Thus the existence of the limit function is proved.

Hence we have

Theorem 13. *Let $f(z)$ belong to $\mathfrak{H}^p_\mu$ $(p\geqq 1,\ 0\leqq\alpha<1)$. Then $f(z)$ is represented by its Cauchy and Poisson integral. As for real part of $f(x)$ we have also*

$$f(z)=2C(z,\Re f)=P(z,\Re f)+i\tilde{P}(z,\Re f). \tag{3.11}$$

Theorem 14. *Let $f(z)$ belong to $\mathfrak{H}^p_\mu$ $(p\geqq 1,\ 0\leqq\alpha<1)$. Then we have*

$$\lim_{y\to 0}\int_{-\infty}^{\infty}\frac{|f(x+iy)-f(x)|^p}{1+|x|^\alpha}dx=0. \tag{3.12}$$

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