149. On Convolution of Laurent Series

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1. Related to a conjecture proposed by Pólya and Schoenberg [4], we have observed in a previous paper [1] a class $\mathfrak{R}_0$ of regular analytic functions defined in the unit circle $|z|<1$ which are of positive real part there and equal to unity at the origin. It has been shown that if both functions

$$f(z)=1+2\sum_{n=1}^{\infty}a_nz^n \quad \text{and} \quad g(z)=1+2\sum_{n=1}^{\infty}b_nz^n$$

belong to $\mathfrak{R}_0$ then the function defined by

$$h(z)=1+2\sum_{n=1}^{\infty}a_nb_nz^n$$

also belongs to $\mathfrak{R}_0$.

In the same paper [1], we have also observed, as a straightforward generalization of the class $\mathfrak{R}_0$, a class $\mathfrak{R}_q$ of single-valued regular analytic functions defined in an annulus $(0<)q<|z|<1$ which are of positive real part and normalized by the conditions that their values on $|z|=q$ have the constant real part and that their Laurent expansions have the constant term equal to unity. For this class, it has been shown that if both functions

$$f(z)=1+2\sum_{n=-\infty}^{\infty}{}' \frac{a_n}{1-q^{2n}}z^n \quad \text{and} \quad g(z)=1+2\sum_{n=-\infty}^{\infty}{}' \frac{b_n}{1-q^{2n}}z^n$$

belong to $\mathfrak{R}_q$ then the function defined by

$$h(z)=1+2\sum_{n=-\infty}^{\infty}{}' \frac{a_nb_n}{1-q^{2n}}z^n$$

also belongs to $\mathfrak{R}_q$; here the prime means that the summand with the suffix $n=0$ is to be omitted.

On the other hand, in a previous paper [2], we have considered, together with the classes mentioned above, a wider class $\hat{\mathfrak{R}}_q$ which is obtained by rejecting the restricting condition for $\mathfrak{R}_q$ imposed on image of $|z|=q$. Namely, the class consists of single-valued regular analytic functions defined in an annulus $(0<)q<|z|<1$ which are of positive real part and normalized by the condition that their Laurent expansions have the constant term equal to unity.

The result on $\mathfrak{R}_q$ referred to above does not admit a formally direct generalization for the class $\hat{\mathfrak{R}}_q$ as it stands. In fact, for functions

$$f(z)=1+2\sum_{n=-\infty}^{\infty}{}' \frac{a_n}{1-q^{2n}}z^n \quad \text{and} \quad g(z)=1+2\sum_{n=-\infty}^{\infty}{}' \frac{b_n}{1-q^{2n}}z^n$$

both belonging to $\hat{\mathfrak{R}}_q$, the function defined by

$$h(z)=1+2\sum_{n=-\infty}^{\infty}{}' \frac{a_n b_n}{1-q^{2n}}z^n$$

is not necessarily even convergent in $q<|z|<1$. For instance, we may observe a particular function

$$\frac{2}{i}\left(\zeta\left(i\lg\frac{q}{z}\right)-\frac{\eta_1}{\pi}i\lg\frac{q}{z}\right)=1-2\sum_{n=-\infty}^{\infty}{}' \frac{q^n}{1-q^{2n}}z^n,$$

the elliptic zeta-function depending on the primitive quasi-periods $2\omega_1=2\pi$ and $2\omega_3=-2i\lg q$. It maps $q<|z|<1$ univalently onto the right half-plane cut along a vertical rectilinear segment with the real part equal to unity and hence belongs surely to $\hat{\mathfrak{R}}_q$. The Laurent series obtained by convoluting it with itself as in the manner described above, namely the series

$$1+2\sum_{n=-\infty}^{\infty}{}' \frac{q^{2n}}{1-q^{2n}}z^n$$

converges if and only if z is contained in the annulus $1<|z|<1/q^2$. Hence, in order to obtain an analogue for $\hat{\mathfrak{R}}_q$ as a generalization of the result established for $\mathfrak{R}_q$, a modification becomes necessary.

The purpose of the present paper is to show that such a modification is actually possible.

2. As shown in [2], any function $\Phi(z)\in\hat{\mathfrak{R}}_q$ can be uniquely decomposed into the form

$$\Phi(z)=R(z)+T(z)-1;\qquad R(z)\in\mathfrak{R}_q,\ T(z)\in\mathfrak{R}'_q$$

where $\mathfrak{R}'_q$ denotes the class consisting of functions $\Psi(z)$ such that $\Psi(q/z)$ belongs to $\mathfrak{R}_q$. Based on this characteristic decomposition, we can establish a result for $\hat{\mathfrak{R}}_q$ which may be stated as follows.

THEOREM. *Let $f(z)$ and $g(z)$ both belong to the class $\hat{\mathfrak{R}}_q$ and their Laurent expansions be given by*

$$f(z)=1+2\sum_{n=-\infty}^{\infty}{}' \frac{a_n-q^n a'_n}{1-q^{2n}}z^n \quad \textit{and} \quad g(z)=1+2\sum_{n=-\infty}^{\infty}{}' \frac{b_n-q^n b'_n}{1-q^{2n}}z^n$$

where, according to the characteristic decompositions, it is supposed that

$$1+2\sum_{n=-\infty}^{\infty}{}' \frac{a_n}{1-q^{2n}}z^n\in\mathfrak{R}_q,\qquad 1-2\sum_{n=-\infty}^{\infty}{}' \frac{q^n a'_n}{1-q^{2n}}z^n\in\mathfrak{R}'_q,$$

$$1+2\sum_{n=-\infty}^{\infty}{}' \frac{b_n}{1-q^{2n}}z^n\in\mathfrak{R}_q,\qquad 1-2\sum_{n=-\infty}^{\infty}{}' \frac{q^n b'_n}{1-q^{2n}}z^n\in\mathfrak{R}'_q.$$

Then the function defined by

$$h(z)=1+2\sum_{n=-\infty}^{\infty}{}' \frac{a_n b_n-q^n a'_n b'_n}{1-q^{2n}}z^n$$

also belongs to $\hat{\mathfrak{R}}_q$.

Proof. In view of the result for $\mathfrak{R}_q$ established in [1], we conclude immediately that the function defined by

$$R(z)=1+2\sum_{n=-\infty}^{\infty}{}' \frac{a_n b_n}{1-q^{2n}} z^n$$

belongs to $\mathfrak{R}_q$. On the other hand, we see by assumption that both

$$1+2\sum_{n=-\infty}^{\infty}{}' \frac{a'_{-n}}{1-q^{2n}} z^n \quad \text{and} \quad 1+2\sum_{n=-\infty}^{\infty}{}' \frac{b'_{-n}}{1-q^{2n}} z^n$$

belong to $\mathfrak{R}_q$. By the same reason as above, the function defined by

$$S(z)=1+2\sum_{n=-\infty}^{\infty}{}' \frac{a'_{-n} b'_{-n}}{1-q^{2n}} z^n$$

belongs to $\mathfrak{R}_q$, and hence the function defined by

$$T(z)=S\left(\frac{q}{z}\right)=1-2\sum_{n=-\infty}^{\infty}{}' \frac{q^n a'_n b'_n}{1-q^{2n}} z^n$$

belongs to $\mathfrak{R}'_q$. Consequently, the function $h(z)$ defined in the theorem is expressed by

$$h(z)=R(z)+T(z)-1; \qquad R(z)\in\mathfrak{R}_q, \ T(z)\in\mathfrak{R}'_q.$$

The right-hand member in the last relation expresses the decomposition of $h(z)$ characteristic to the class $\hat{\mathfrak{R}}_q$, whence follows the assertion of the theorem.

Finally we state a supplementary remark: The decomposition theorem on $\hat{\mathfrak{R}}_q$ referred to above, combined with an integral representation for $\mathfrak{R}_q$, or, equivalently and rather directly, an integral representation for $\hat{\mathfrak{R}}_q$ itself yields readily an integral representation of Laurent coefficients of a function from the class $\hat{\mathfrak{R}}_q$. In fact, the coefficients of $g(z)$ in the theorem are given by

$$b_n=\int_{-\pi}^{\pi} e^{-in\varphi}d\rho(\varphi), \quad b'_n=\int_{-\pi}^{\pi} e^{-in\varphi}d\tau(\varphi)$$

where $\rho(\varphi)\equiv\rho_g(\varphi)$ and $\tau(\varphi)\equiv\tau_g(\varphi)$ are real-valued increasing functions associated to $g(z)$ which are defined for $-\pi<\varphi\leqq\pi$ and with the total variation equal to unity; cf. [3]. It is then readily verified that the expression of $h(z)$ can be transformed into

$$h(z)=\int_{-\pi}^{\pi} R_1(ze^{-i\varphi})d\rho(\varphi)+\int_{-\pi}^{\pi} T_1(ze^{-i\varphi})d\tau(\varphi)-1$$

where $R_1(z)$ and $T_1(z)$ denote the components of $f(z)$ in its characteristic decomposition. Since the first and second terms in the last expression of $h(z)$ belong evidently to the classes $\mathfrak{R}_q$ and $\mathfrak{R}'_q$, respectively, we conclude again that $h(z)$ surely belongs to the class $\hat{\mathfrak{R}}_q$.

The alternative proof of the theorem just mentioned may be regarded certainly as a modification of the argument previously employed for establishing the result in the case of $\mathfrak{R}_q$.

References

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