17. Convergence Concepts in Semi-ordered Linear Spaces. II

By Hidegorô NAKANO
Hokkaido University

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In the part I*) we discussed the standard modificators in the case where R is super-universally continuous, and we obtained Theorems 3 and 4. In the sequel, these theorems will be extended to more general cases which are essentially important in the theory of semi-ordered linear spaces.

An operator $\mathfrak{a}$ is said to be *reducible*, if $(Pa_\nu)^{\mathfrak{a}}=Pa_\nu^{\mathfrak{a}}$ $(\nu=0,1,2,\cdots)$ for every projection operator P on R. A modificator A is said to be *reducible*, if every operator of A is reducible. All sub., loc. and ind. operators are obviously reducible, and hence S, L, I and all standard modificators are reducible. We see easily that AB and $A\circ B$ are reducible, if both A and B are reducible. Every reducible modificator commutes evidently all loc. operators by definition.

A semi-ordered linear space R is said to be *locally super-universally continuous*, if R is continuous and we can find a system of projectors $[p_\lambda]$ $(\lambda\in\Lambda)$ such that $\bigcup_{\lambda\in\Lambda}[p_\lambda]=1$ and $[p_\lambda]R$ is super-universally continuous for all $\lambda\in\Lambda$.

Lemma 5. *If R is locally super-universally continuous, then we have*

$$ALSB\succ LASLB$$

for every two reducible modificators A and B.

Proof. Let $[p_\lambda]$ $(\lambda\in\Lambda)$ be a system of projectors such that $\bigcup_{\lambda\in\Lambda}[p_\lambda]=1$ and all $[p_\lambda]R$ $(\lambda\in\Lambda)$ are super-universally continuous. Recalling Lemma 4, we have $ALSB\succ ASLB$ in $[p_\lambda]R$ for every $\lambda\in\Lambda$. Thus we have in R

$$ALSB\succ LALSB\succ LASLB.$$

Lemma 6. *If R is locally super-universally continuous, then*

$$(L\circ S)(L\circ S)\sim SLS.$$

Proof. As $L\circ S\geqq LS$ by (2), we have by (3)

$$(L\circ S)(L\circ S)\geqq(L\circ S)LS.$$

We suppose $a_0=(L\circ S)LS\text{-}\lim_{\nu\to\infty}a_\nu$. Then, by virture of Theorem 1, we can find $\mathfrak{L}_0\in L$ and $\mathfrak{S}_0\in S$ such that

$$a_0^{\mathfrak{l}\mathfrak{s}}=LS\text{-}\lim_{\nu\to\infty}a_\nu^{\mathfrak{l}\mathfrak{s}}\quad\text{for all }\mathfrak{l}\in\mathfrak{L}_0,\ \mathfrak{s}\in\mathfrak{S}_0.$$

As R is locally super-universally continuous, we can suppose here that

*) H. Nakano and M. Sasaki: Convergence concepts in semi-ordered linear spaces. I, Proc. Japan Acad., **35**, no. 1 (1959).

$[p]R$ is super-universally continuous for every $\mathfrak{l}[p]\in\mathfrak{L}_0$. Then we have by Lemma 4

$$a_0^{\mathfrak{l}\mathfrak{s}}=L\circ S\text{-}\lim_{\nu\to\infty} a_\nu^{\mathfrak{l}\mathfrak{s}} \quad \text{for all } \mathfrak{l}\in\mathfrak{L}_0,\ \mathfrak{s}\in\mathfrak{S}_0,$$

and hence $a_0=(L\circ S)(L\circ S)\text{-}\lim_{\nu\to\infty} a_\nu$. Thus we have $(L\circ S)(L\circ S)\prec(L\circ S)LS$ by definition. We conclude therefore $(L\circ S)(L\circ S)\sim(L\circ S)LS$. On the other hand we have by (2), (3), (4)

$$SLS=SLLS\leqq(L\circ S)LS\leqq L\circ(SLS)\prec SLS.$$

A modificator is said to be *simple*, if it is composed from S, L, I and $(L\circ S)$ only by product. Simple modificators are obviously standard. It is so complicated to discuss standard modificators in general. Thus we consider here only simple modificators.

Theorem 5. *If R is locally super-universally continuous, then every simple modificator is equivalent to one of*

$$LSL\prec SLS\prec {LS \atop SL}\prec L\circ S\prec {L \atop S}\prec O.$$

Proof. In general we have

$$(17) \qquad (L\circ S)I\sim I(L\circ S)\sim SL.$$

Because we have obviously

$$(L\circ S)I=L\circ S\circ I=L\circ(SI)=(SI)\circ L,$$

and by (12), (6), (7), (14)

$$SL\sim SI\succ(SI)\circ L\geqq SIL\sim SL$$

and furthermore by (6), (7), (16)

$$SL\sim SI\sim IS\succ I(L\circ S)=I(S\circ L)\geqq ISL.$$

Here we have

$$(18) \qquad ISL\sim SL.$$

Because we have by Lemma 3, (12), (16), (11)

$$SL\succ ISL\sim ISI\sim IIS=IS\sim SI\sim SL.$$

As we have by (7), (4), Lemma 3

$$LS\succ(LS)\circ L\geqq L(S\circ L)=L(L\circ S)\geqq LLS=LS,$$
$$LS\succ S\circ(LS)\geqq(S\circ L)S=(L\circ S)S\geqq LSS=LS,$$
$$LS\succ(LS)\circ S\geqq LSS=LS,$$

we obtain

$$(19) \quad L(L\circ S)\sim(L\circ S)S\sim L\circ(LS)\sim(LS)\circ L\sim S\circ(LS)\sim(LS)\circ S\sim LS.$$

As we have by (7), (4), Lemma 3

$$SL\succ L\circ(SL)\geqq(L\circ S)L=(S\circ L)L\geqq SLL=SL,$$
$$SL\succ(SL)\circ S\geqq S(L\circ S)=S(S\circ L)\geqq SSL=SL,$$
$$SL\succ S\circ(SL)\geqq SSL=SL,$$

we obtain

$$(20) \quad S(L\circ S)\sim(L\circ S)L\sim L\circ(SL)\sim(SL)\circ L\sim S\circ(SL)\sim(SL)\circ S\sim SL.$$

Now we suppose that R is locally super-universally continuous. Putting $A=S$, $B=O$ in Lemma 5, we obtain

$$SLS\succ LSSL=LSL.$$

Thus we have by (7) and Lemma 3

$$LSL \prec SLS \prec {}^{LS}_{SL} \prec L\circ S \prec {}^{L}_{S} \prec O.$$

We need only to prove that for each one C of these modificators, each one of LC, SC, IC, $(L\circ S)C$ also is equivalent to one of them.

For $C=LSL$, we have obviously $LLSL=LSL$ by (9). Putting $A=S$, $B=L$ in Lemma 5, we obtain by Lemma 3

$$LSL \succ S\circ(LSL) \geqq SLSL \succ LSSLL = LSL,$$

and hence $SLSL \sim LSL$. Putting $A=I$, $B=L$ in Lemma 5, we obtain by (18)

$$LSL \succ ILSL \succ LISLL = LISL \sim LSL,$$

and hence $ILSL \sim LSL$. As LSL is regular by Lemma 3, we also have

$$LSL \succ (L\circ S)LSL \geqq LSLSL = L(SLSL) \sim LLSL = LSL$$

and hence $(L\circ S)LSL \sim LSL$.

For $C=SLS$, putting $A=LS$, $B=O$ in Lemma 5, we obtain by (7)

$$LSL \succ LSLS \succ LLSSL = LSL,$$

and hence $LSLS \sim LSL$. Putting $A=IS$, $B=O$ in Lemma 5, we obtain by (18)

$$ISLS \succ LISSL = LISL \sim LSL.$$

On the other hand we have by (2), (12)

$$ISLS \leqq I\circ(SLS) = SLSI \sim SLSL \sim LSL.$$

Thus we have $ISLS \sim LSL$. As we have by (2), (4)

$$LSLS = LSSLS \leqq (L\circ S)SLS \leqq S\circ(LSLS) \prec LSLS,$$

we also obtain $(L\circ S)SLS \sim LSL$.

For $C=SL$, we see easily by (2), (4), (18)

$$LC \sim (L\circ S)C \sim LSL, \quad IC \sim SC \sim C.$$

For $C=LS$, we see easily by (2), (4)

$$SC \sim (L\circ S)C \sim SLS, \quad LC \sim C.$$

Putting $A=I$, $B=O$ in Lemma 5, we obtain by (18)

$$ILS \succ LISL \sim LSL.$$

On the other hand we have by (2), (12), (16)

$$ILS \leqq (I\circ L)S = LIS \sim LSI \sim LSL.$$

Thus we obtain $IC \sim LSL$.

For $C=L\circ S$, we have obviously by (17), (19), (20), Lemma 6

$$LC \sim LS, \quad SC \sim IC \sim SL, \quad (L\circ S)C \sim SLS.$$

For $C=L$ or S, we need not discuss, because it is trivial by (19), (20).

Theorem 6. *If R is locally super-universally continuous and complete, then every standard modificator is equivalent to one of*

$$LS \prec SL \prec {}^{L}_{S} \prec O.$$

Proof. Let $[p_\lambda]$ $(\lambda\in\Lambda)$ be a system of projectors such that $\bigcup_{\lambda\in\Lambda}[p_\lambda]=1$ and $[p_\lambda]R$ is super-universally continuous for all $\lambda\in\Lambda$. As $L\sim O$ in $[p_\lambda]R$, we have $SL\sim S$ in $[p_\lambda]R$ for every $\lambda\in\Lambda$. Thus we have

$LSL \sim LS$ in R. Therefore we conclude $LSL \sim SLS \sim LS$ by Theorem 5.

If $a_0 = SL\text{-}\lim_{\nu\to\infty} a_\nu$, then we can find $\mathfrak{S} \in S$ by definition such that

$$a_0^{\mathfrak{s}} = L\text{-}\lim_{\nu\to\infty} a_\nu^{\mathfrak{s}} \quad \text{for every } \mathfrak{s} \in \mathfrak{S}.$$

As $L \sim O$ in $[p_\lambda]R$, we obtain hence

$$([p_\lambda][p]a_0)^{\mathfrak{s}} = \lim_{\nu\to\infty} ([p_\lambda][p]a_\nu)^{\mathfrak{s}} \quad \text{for every } \mathfrak{s} \in \mathfrak{S}, \ \lambda \in \Lambda, \ p \in R.$$

Thus, putting $\mathfrak{L} = \{[p_\lambda][p]: \lambda \in \Lambda, \ p \in R\}$, we have $\mathfrak{L} \in L$ and

$$a_0^{\mathfrak{l}\mathfrak{s}} = \lim_{\nu\to\infty} a_\nu^{\mathfrak{l}\mathfrak{s}} \quad \text{for } \mathfrak{l} \in \mathfrak{L}, \ \mathfrak{s} \in \mathfrak{S},$$

and hence $a_0 = L\circ S\text{-}\lim_{\nu\to\infty} a_\nu$ by definition. Thus we have $SL \succ (L\circ S)$, and consequently $SL \sim (L\circ S)$ by (2). Therefore we conclude by Theorem 5 that every simple modificator is equivalent to one of

$$LS \prec SL \prec \frac{L}{S} \prec O.$$

Now we can prove that every standard modificator is equivalent to one of them. For this, we need only to show that for every pair C_1, C_2 of them, $C_1 \circ C_2$ is equivalent to one of them. First of all, we have $L\circ S \sim SL$, as proved just above. By virtue of (19) and (20), we have obviously

$$L\circ(LS) \sim (LS)\circ L \sim (LS)\circ S \sim S\circ(LS) \sim LS,$$
$$L\circ(SL) \sim (SL)\circ L \sim (SL)\circ S \sim S\circ(SL) \sim SL.$$

We also have by (4), (2), Lemma 3

$$SL \succ (SL)\circ(SL) \geqq S(L\circ(SL)) = S((SL)\circ L) \geqq SSLL = SL,$$
$$LS \succ (LS)\circ(SL) \geqq LSSL = LSL \sim LS,$$
$$LS \succ (SL)\circ(LS) \geqq SLLS = SLS \sim LS,$$
$$LS \succ (LS)\circ(LS) \geqq LSLS \sim LLS = LS,$$

and hence $(SL)\circ(SL) \sim SL$, $(LS)\circ(SL) \sim (SL)\circ(LS) \sim (LS)\circ(LS) \sim LS$.

Example 1. Let $\mathfrak{S}$ be a totally additive set class on a space S; $m(E)$ $(E \in \mathfrak{S})$ a totally additive measure; and R_0 the totality of measurable functions on S. For φ, $\psi \in R_0$ we define $\varphi \geqq \psi$, if

$$m\{x: \varphi(x) < \psi(x), \ x \in E\} = 0 \quad \text{for } m(E) < +\infty,$$

that is, $\varphi(x) \geqq \psi(x)$ almost everywhere in E for $m(E) < +\infty$. Then we see easily that R_0 constitutes a locally super-universally continuous, complete, semi-ordered linear space. Let R_1 be the set of all such measurable functions φ on S that we can find a sequence of sets $E_\nu \in \mathfrak{S}$ with $m(E_\nu) < +\infty$ $(\nu = 1, 2, \cdots)$ for which $x \bar{\in} E_\nu$ for all $\nu = 1, 2, \cdots$ implies $\varphi(x) = 0$. R_1 is obviously a semi-normal manifold of R_0 and we see easily that R_1 is super-universally continuous and complete. We denote by R an arbitrary semi-normal manifold of R_0. R is obviously locally super-universally continuous.

There are two well-known convergence concepts in R, that is, the point convergence and the measure convergence. A sequence $\varphi_\nu \in R$ $(\nu = 1, 2, \cdots)$ is said to be *point convergent* to φ_0, if $\lim_{\nu\to\infty} \varphi_\nu(x) = \varphi_0(x)$ almost everywhere in E for $m(E) < +\infty$. A sequence $\varphi_\nu \in R$ $(\nu = 1, 2,$

$\cdots$) is said to be *measure convergent* to φ_0, if

$$\lim_{\nu\to\infty} m\{x\colon |\varphi_\nu(x)-\varphi_0(x)|<\varepsilon,\ x\in E\}=0 \quad \text{for } \varepsilon>0,\ m(E)<+\infty.$$

We can prove easily: *the point convergence is equivalent to L-convergence in* R_0, *O-convergence in* R_1, *and L-convergence in* R; *the measure convergence is equivalent to LS-convergence in* R_0, *S-convergence in* R_1, *and LSL-convergence in* R.

Example 2. Let M be the so-called M-space on the closed interval $[0, 1]$, that is, M consists of all bounded measurable functions on $[0, 1]$ and $\varphi\geqq\psi$ is defined as $\varphi(x)\geqq\psi(x)$ almost everywhere in $[0, 1]$ for the Lebesgue measure. M is super-universally continuous. For each pair of natural numbers $\mu\leqq\nu$, we denote by $\chi_{(\mu,\nu)}$ the characteristic function of the closed interval $\left[\frac{\mu-1}{\nu}, \frac{\mu}{\nu}\right]$. As the set of all pairs (μ, ν) is countable, we can consider $\{\chi_{(\mu,\nu)}\}_\nu$ a sequence. Then we have obviously

$$\overline{\lim_{\nu\to\infty}}\, \chi_{(\mu,\nu)}(x)=1, \qquad \underline{\lim_{\nu\to\infty}}\, \chi_{(\mu,\nu)}(x)=0$$

for every point x in $[0, 1]$. For a sub. operator $\mathfrak{s}$, if $\left\{\left(\frac{\mu}{\nu}\right)^{\mathfrak{s}}\right\}_\nu$ is convergent, then $\lim_{\nu\to\infty} \chi^{\mathfrak{s}}_{(\mu,\nu)}(x)=0$ except for $x=\lim_{\nu\to\infty}\left(\frac{\mu}{\nu}\right)^{\mathfrak{s}}$. Thus $\lim_{\nu\to\infty}\chi^{\mathfrak{s}}_{(\mu,\nu)}=0$ in M, because $\{\chi_{(\mu,\nu)}\}_\nu$ is bounded. Therefore we have

$$S\text{-}\lim_{\nu\to\infty} \chi_{(\mu,\nu)} = 0.$$

We have obviously for every point x in $[0, 1]$

$$\overline{\lim_{\nu\to\infty}}\, \nu\chi_{(\mu,\nu)}(x)=+\infty, \qquad \underline{\lim_{\nu\to\infty}}\, \nu\chi_{(\mu,\nu)}(x)=0.$$

For a sub. operator $\mathfrak{s}$, if $\left\{\left(\frac{\mu}{\nu}\right)^{\mathfrak{s}}\right\}_\nu$ is convergent, then $\lim_{\nu\to\infty}(\nu\chi_{(\mu,\nu)})^{\mathfrak{s}}(x)=0$ except for $x=\lim_{\nu\to\infty}\left(\frac{\mu}{\nu}\right)^{\mathfrak{s}}$, and hence

$$L\text{-}\lim_{\nu\to\infty} (\nu\chi_{(\mu,\nu)})^{\mathfrak{s}} = 0$$

but not O-convergent, because $\{(\nu\chi_{(\mu,\nu)})^{\mathfrak{s}}\}_\nu$ is not bounded in M. Thus

$$SL\text{-}\lim_{\nu\to\infty} \nu\chi_{(\mu,\nu)} = 0,$$

but $\{\nu\chi_{(\mu,\nu)}\}$ *is not LS-convergent.*

Let $\mathfrak{S}_0$ be the totality of sub. operators. We denote by R the set of all mappings from $\mathfrak{S}_0$ into M. For each $x\in R$, denoting by $x(\mathfrak{s})$ the image of $\mathfrak{s}\in\mathfrak{S}_0$ by x, we define $\alpha x+\beta y$ for $x, y\in R$ as

$$(\alpha x+\beta y)(\mathfrak{s})=\alpha x(\mathfrak{s})+\beta y(\mathfrak{s}) \qquad \text{for all } \mathfrak{s}\in\mathfrak{S}_0,$$

and $x\geqq y$ as $x(\mathfrak{s})\geqq y(\mathfrak{s})$ for all $\mathfrak{s}\in\mathfrak{S}_0$. Then we see easily that R is universally continuous and locally super-universally continuous, and for a sequence $\{x_\nu\}_{\nu\geqq 1}$ we have $\lim_{\nu\to\infty} x_\nu=0$ in R if and only if $\lim_{\nu\to\infty} x_\nu(\mathfrak{s})=0$ in M for all $\mathfrak{s}\in\mathfrak{S}_0$.

We can find uniquely a sequence $u_\nu\in R$ $(\nu=1, 2, \cdots)$ such that for every sub. operator $\mathfrak{s}\{\mu_1, \mu_2, \cdots\}$ we have

$$\{u_\nu(\mathfrak{s})^{\mathfrak{s}}\}_\nu = \{\chi_{(\mu,\nu)}\}_\nu, \quad u_\nu(\mathfrak{s}) = 0 \quad \text{for } \nu \neq \mu_\rho \ (\rho = 1, 2, \cdots).$$

As to this sequence $\{u_\nu\}_{\nu\geq 1}$, we see easily that $LS\text{-}\lim_{\nu\to\infty} u_\nu = 0$ *but* $\{u_\nu\}_\nu$ *is not SL-convergent;* $LSL\text{-}\lim_{\nu\to\infty} \nu u_\nu = 0$ *but* $\{\nu u_\nu\}_{\nu\geq 1}$ *is not SLS-convergent.*

We also can find uniquely a sequence $v_\nu \in R$ $(\nu = 1, 2, \cdots)$ such that

$$\{v_\nu(\mathfrak{s})\}_\nu = \{\chi_{(\mu,\nu)}\}_\nu \quad \text{for all } \mathfrak{s} \in \mathfrak{S}_0.$$

As to this sequence $\{v_\nu\}_{\nu\geq 1}$ we have $S\text{-}\lim_{\nu\to\infty} v_\nu = 0$, $SL\text{-}\lim_{\nu\to\infty} \nu v_\nu = 0$ in R *but* $\{\nu v_\nu\}_{\nu\geq 1}$ *is not LS-convergent.* We see easily furthermore that

$$SLS\text{-}\lim_{\nu\to\infty} (u_\nu + \nu v_\nu) = 0,$$

but $\{u_\nu + \nu v_\nu\}_{\nu\geq 1}$ *is neither LS- nor SL-convergent.*