28. *On Minimal Slit Domains*

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An arbitrary plane domain of finite connectivity is mapped conformally onto the whole plane with the slits parallel to the real axis, and the mapping is uniquely determined except some suitable linear transformations. But when the domain is of infinite connectivity, the uniqueness does not hold. Koebe introduced the concept *minimalen Schlitzbereich* as the normal domain, the mapping onto which is unique (cf. [1, 2]). This mapping function is obtained as the solution of an extremal problem (cf. [3]). In this paper, we deal with the existence of the extremal mapping function onto the circular ring with radial or circular slits as normal domain and some properties of this normal domain, *minimal ring*.

1. Conformal mapping onto a circular ring with the radial slits

Let Ω be a plane domain bounded by $N+2$ analytic Jordan curves $\gamma_1, \gamma_2, \delta_1, \cdots, \delta_N$. We define the class $\mathfrak{H}$ of function h on Ω, satisfying the following conditions:

1) h is harmonic in Ω,

2) there is a negative constant $k(h)$ depending on h, and $h=k(h)$ on γ_2, and $h=0$ on γ_1,

3) for the conjugate harmonic function h^* of h,

$$\int_{\gamma_1} dh^*=2\pi, \quad \int_{\gamma_2} dh^*=-2\pi, \quad \int_{\delta_j} dh^*=0 \qquad (j=1,\cdots,N).$$

Let h_0 be the function of $\mathfrak{H}$ such that

$$\frac{\partial h_0}{\partial n}=0 \quad \text{on } \delta_j \qquad (j=1,\cdots,N).$$

Then, the function $f(z)=e^{h_0+ih_0^*}$ maps Ω onto a circular ring $r_0<|w|<1$ $(r_0=k(h_0))$ with N radial slits in the w-plane.

Lemma 1. *The function h_0 minimizes the functional*

$$2\pi k(h)+\int_{\delta} h\, dh^* \qquad (\delta=\bigcup_j \delta_j)$$

in $\mathfrak{H}$ and the minimizing function is unique.

Proof.
$$D(h-h_0)=\int_{\partial\Omega}(h-h_0)\, d(h-h_0)^*$$
$$=\int_{\delta} h\, dh^*+\int_{\delta}(h\, dh_0^*-h_0\, dh^*)$$

$$=\int_{\delta} h\,dh^* + \int_{-\gamma_1-\gamma_2} (h\,dh_0^* - h_0\,dh^*)$$
$$=\int_{\delta} h\,dh^* + 2\pi(k(h)-k(h_0)).$$

Hence we have

$$2\pi k(h)+\int_{\delta} h\,dh^* = 2\pi k(h_0)+D(h-h_0). \tag{1}$$

Since $D(h-h_0)\geqq 0$, the left member in (1) attains its minimum when $h=h_0$. From the condition 2) follows the uniqueness of such function. (Q. E. D.)

In the next place, we shall denote by the same notation Ω the general domain bounded by two analytic Jordan curves γ_1, γ_2 and a closed set δ encircled by γ_1 and γ_2 such that $\delta\frown\gamma_1=0$ and $\delta\frown\gamma_2=0$. Consider the monotonic exhaustion Ω_n of Ω, the boundary of which consists of a finite number of analytic Jordan curves $\gamma_1, \gamma_2, \delta_1, \cdots, \delta_{N(n)}$. We define the class $\mathfrak{F}_n$ (resp. $\mathfrak{F}$) of functions on Ω_n (resp. Ω) as follows: $f\in\mathfrak{F}_n$ (resp. $\mathfrak{F}$) is regular and univalent in Ω_n (resp. Ω), $|f(z)|=1$ on γ_1, $|f(z)|=r(f)<1$ on γ_2, and $f(z_0)=1$ for a fixed point z_0 on γ_1. The logarithmic area $A(f)$ of the "image" of δ by $f\in\mathfrak{F}$ is defined as the limit as $\Omega_n\to\Omega$ of the logarithmic area of the domain enclosed by the image δ_j' of δ_j.

Theorem 1. *There exists a unique function $f_0\in\mathfrak{F}$ such that*

$$\operatorname{Min}\{2\pi\log r(f)-A(f)\}=2\pi\log r(f_0).$$

The function f_0 maps the domain Ω conformally onto a concentric circular ring with the radial slits whose area measure is zero.

Proof. $h=\log f$ belongs to $\mathfrak{H}$ for $f\in\mathfrak{F}$. Applying Lemma 1 to the functional $L_n(f,\Omega_n)=2\pi\log r(f)-A_n(f)$ of $\mathfrak{F}_n$, we have the function f_n minimizing this functional in $\mathfrak{F}_n$. For $n\leqq m$, $L_n(f,\Omega_n)\leqq L_m(f,\Omega_m)$. Since $\{f_n\}$ is the normal family, the limiting function f_0 belongs to $\mathfrak{F}$. By the reduction theorem (cf. [3]) the existence of the extremal function f_0 is shown. The deviation of an arbitrary function f of $\mathfrak{F}$ from this extremum is $D(\log|f/f_0|)$. From the normalization condition on z_0, we obtain the uniqueness of such extremal function. The equality shows that $A(f_0)=0$.

Suppose that there is a connected component d_1 of "image" of δ which is not radial. Let $w'=\varphi(w)$ map conformally a triply connected domain bounded by Γ_1: $|w|=1$, Γ_2: $|w|=r_0$ and d_1 onto a circular ring with radial slit as image of d_1. By Lemma 1 we have $r'<r_0$. Since $F(z)=\varphi(f_0(z))$ is a function of $\mathfrak{F}$, this contradicts the extremal property of f_0.

2. Minimal circular ring

Let G be the image of Ω by the extremal function f_0 and $\mathfrak{K}$ the class of piecewise smooth functions $g=g(w)$ in G which vanish on Γ_1, Γ_2, and $D(g)<\infty$. Each G has the following property:

(*) $\qquad D(\log|w|, g)=0 \quad$ for arbitrary $g \in \mathfrak{K}$.

This condition can be replaced by

(**) $\qquad D(\log|w|+g) \geqq D(\log|w|) \quad$ for arbitrary $g \in \mathfrak{K}$.

In general, a circular ring with the radial slits satisfying the condition (*) or (**) is called a *minimal ring* (with the radial slits).

Theorem 2. *If $w'=\varphi(w)$ is a conformal mapping from a minimal ring G onto a minimal ring G' satisfying $\varphi(1)=1$, then $\varphi(w)\equiv w$.*

Proof. We set $g(w)=\log|w/\varphi(w)|$. Let C be an arbitrary closed curve in G separating Γ_2 from other boundaries, and σ_i the subdomain bounded by C and Γ_i $(i=1,2)$. These images are denoted by C', Γ_1', Γ_2', σ_1' and σ_2' respectively. Let g_1 be the smooth function which coincides with g in σ_1 and vanishes on Γ_2. The function g_1 belongs to $\mathfrak{K}$. Setting $g_2=g-g_1$, from the minimal property of G and G'

$$D(\log|w|, g)=D(\log|w|, g_2)=\int_{\Gamma_2} g_2 d\theta=2\pi\log r/r'$$

$$D(\log|\varphi(w)|, g)=D(\log|w'|, g_2)=\int_{\Gamma_2'} g_2 d\theta'=2\pi\log r/r'.$$

Hence we have

$$D(\log|w/\varphi(w)|)=D(\log|w|-\log|\varphi(w)|, g)=0.$$

By the condition $\varphi(1)=1$ we have $\varphi(w)\equiv w$. (Q. E. D.)

Let G be an arbitrary minimal ring and $\{G_n\}$ its exhaustion. Let f_n the function which maps conformally G_n onto a circular ring with the radial slits. The limiting function f_0 maps G conformally onto a minimal ring, and by Theorem 2 we have $f_0(w)\equiv w$.

Theorem 3. *The area measure of the set of radial slits of a minimal ring is zero.*

Theorem 4. *Let G be the circular ring $r<|w|<1$ with radial slits, $\{C\}$ be a family of rectifiable curves in G binding Γ_1 and Γ_2, and λ_c the extremal length of $\{C\}$. The necessary and sufficient condition for G to be minimal is*

$$\lambda_c=(2\pi)^{-1}\log 1/r.$$

Proof. 1° (Necessarity). Let $\{G_n\}$ be an exhaustion of G, $\{C\}_n$ a family of curves in G_n binding Γ_1 and Γ_2, λ_n the extremal length of $\{C\}_n$. Then $\lambda_n\to\lambda$. If we map G_n conformally onto a minimal ring G_n', $r_n<|w|<1$, then, from the conformal invariance of the extremal length and the finite connectivity of G_n, we obtain the corresponding extremal length $\lambda_n=(2\pi)^{-1}\log 1/r_n$. If G is minimal then $r_n\to r$, and therefore we obtain $\lambda=(2\pi)^{-1}\log 1/r$.

2° (Sufficiency). We map G_n onto G_n' by the extremal function f_n. Let $d_1, d_2, \cdots, d_{N(n)}$ be the radial segments binding Γ_1' and Γ_2' which involve the images of $\delta_1, \cdots, \delta_{N(n)}$. Let $G_k'^{(n)}$ be the domain bounded by d_{k-1}, d_k, Γ_1' and Γ_2', $\{C'\}_k^{(n)}$ the family of rectifiable curves in $G_k'^{(n)}$ binding Γ_1' and Γ_2', $\lambda_k^{(n)}$ the corresponding extremal length and $G_k^{(n)}$ the image of $G_k'^{(n)}$ by $f_n^{-1}(w')$. If we map conformally $G_k^{(n)}$ onto a domain $B_k^{(n)}$ in the ζ-plane bounded by two radial segments and two circular arcs $|\zeta|=1$ and $|\zeta|=r$, by the function $\zeta=\phi(w)$ the module of $B_k^{(n)}$ is equal to $\lambda_k^{(n)}$. The Dirichlet integrals of G_n, $G_k^{(n)}$ and $B_k^{(n)}$ are denoted by $D_n(\ \ )$, $D_k^n(\ \ )$ and $D_k^n[\ \ ]$ respectively.

Since the function g of the class $\mathfrak{K}$ belongs to $\mathfrak{K}_{G_n}$, we have by the harmonicity of $\log|\zeta|$

$$\begin{aligned} D_k^{(n)}(\log|w|+g) &= D_k^{(n)}[\log|\phi^{-1}(\zeta)|+g] \\ &\geqq D_k^{(n)}[\log|\zeta|] \\ &= \lambda_k^{(n)-1}(\log 1/r)^2 \end{aligned}$$

Since $\lambda_n^{-1}=\sum_k \lambda_k^{(n)-1}$, we have

$$\begin{aligned} D(\log|w|+g) &= \lim_n D_n(\log|w|+g) \\ &= \lim_n \sum_k D_k^{(n)}(\log|w|+g) \\ &\geqq (\log 1/r)^2 \lim_n \sum_k \lambda_k^{(n)-1} \\ &= (\log 1/r)^2 \lim_n \lambda_n^{-1} \\ &= \lambda^{-1}(\log 1/r)^2 = D(\log|w|). \qquad \text{(Q. E. D.)} \end{aligned}$$

Theorem 5. *Let G be a circular ring with radial slits, and M the projection of the radial slits on the circle $|w|=1$. If the angular measure of M is zero, then G is minimal.*

3. Conformal mapping onto a circular ring with the circular slits

In the case of circular slits we have the results analogous to that of radial slits.

Lemma 2. *Let h_1 be the function of $\mathfrak{H}$ taking the constant values $k_j(h)$ on δ_j. The function h_1 maximizes the functional*

$$2\pi k(h) - \int_\delta h\, dh^*$$

in $\mathfrak{H}$, and the maximizing function is unique.

Theorem 6. *There exists a unique function $f_1 \in \mathfrak{F}$ such that*

$$\operatorname{Max}\{2\pi \log r(f) + A(f)\} = 2\pi \log r(f_1).$$

The function f_1 maps the domain Ω conformally onto a concentric circular ring with the circular slits whose measure is zero.

A circular ring with the circular slits is said to be minimal if $D(\arg w, g)=0$ for an arbitrary $g \in \mathfrak{K}$.

Theorem 7. *The necessary and sufficient condition for G to be minimal is that the extremal length λ_T of the rectifiable closed curves separating Γ_1 and Γ_2 is equal to $2\pi(\log 1/r)^{-1}$*

4. Counterexamples

The converses of Theorems 3 and 5 are not true. That is, there exists the minimal ring such that the angular measure of the projection M of slits is positive (Ex. 1), and there exists the domain which is not minimal, while the area measure of slits is zero (Ex. 2).

Example 1. Consider in the ring $r<|z|<1$ the set of circular slits,

$$\sigma_k(N;q):\ |z|=\frac{r+1}{2},\quad \left|\arg z-\frac{2k\pi}{N}\right|\leqq\frac{q\pi}{N}\qquad (k=1,2,\cdots,N),$$

for $0<q<1$ and a positive integer N. Let S be the domain bounded by Γ_1: $|z|=1$, Γ_2: $|z|=r$ and $\sigma_k(N;q)$.

Lemma. *Let $\{C\}$ be a family of the rectifiable curves binding Γ_1 and Γ_2 in S, λ the extremal length of $\{C\}$. For fixed q, $\lambda\rightarrow(2\pi)^{-1}\log 1/r$ as $N\rightarrow\infty$.*

Proof. Let C_k be the closed curve in S encircling $\sigma_k(N;q)$, F_k the domain bounded by σ_k and C_k. We map S conformally onto the domain S' in the w-plane bounded by Γ_1': $|w|=1$, Γ_2': $|w|=r_N$ and the set of the radial slits

$$\sigma_k'(N;q):\ \left|w-\frac{r_N+1}{2}\right|\leqq\frac{L}{2},\quad \arg w=\frac{2k\pi}{N}\qquad (k=1,2,\cdots,N).$$

We denote the images of C_k and F_k by C_k' and F_k' respectively. We can make the module $M=M(F_k)$ of F_k be constant as $N\rightarrow\infty$. Since the length of the closed curves in F_k' separating C_k' and σ_k' are at least $2L$, the area of F_k', $I(F_k')\geqq 4M(F_k')L^2$. Therefore,

$$L^2\leqq\frac{1}{4N}\sum_k\left(I(F_k')\Big/M(F_k')\right)<\frac{1}{4NM}I(S')<\frac{1}{4NM}\pi.$$

Hence we have $L\rightarrow 0$ as $N\rightarrow\infty$.

Let $\{\gamma'\}$ be the family of the rectifiable closed curves in S' separating Γ_1' and Γ_2'. Since the extremal length of $\{\gamma'\}$ is equal to $2\pi(\log 1/r)^{-1}$, we can make r_N be arbitrarily near to r. Hence, $\lambda=(2\pi)^{-1}\log 1/r_N\rightarrow(2\pi)^{-1}\log 1/r$ as $N\rightarrow\infty$. (Q. E. D.)

Let ε_n and $q_n<1$ $(n=1,2,\cdots)$ be positive numbers, such that $\varepsilon_n\rightarrow 0$ as $N\rightarrow\infty$ and $\prod_n q_n=p>0$. Consider the set of slits $\sigma_k(N_1;q_1)$ in the ring $r<|z|<1$, where N_1 is taken so large that the extremal length of G is $<(1+\varepsilon_1)(2\pi)^{-1}\log 1/r$. This system of slits is denoted by Σ_1.

Next, we consider the system of slits Σ_2;

$$\sigma_k(N_1,N_2;q_1,q_2):\ |z|=\frac{r+1}{2}$$

$$\left|\arg z-\left(\frac{2k_1\pi}{N_1}+\frac{2k_2\pi}{N_1\cdot N_2}\right)\right|\leqq\frac{(r+1)q_1q_2\pi}{2N_1\cdot N_2}\qquad (k_i=1,2,\cdots,N_i),$$

where N_2 is taken so large that $\lambda<(1+\varepsilon_2)(2\pi)^{-1}\log 1/r$.

Thus we define the systems $\Sigma_1,\Sigma_2,\cdots$, successively. Sum of the length of slits tends to $\pi(r+1)\prod_n q_n=\pi(r+1)p>0$, and λ to $(2\pi)^{-1}\log 1/r$. By Theorem 4 the limiting domain is minimal, while the angular measure of projection is positive.

Example 2. We consider the system of slits in the ring $r<|z|<1$,

$$s\colon \arg z=0 \qquad \left||z|-\frac{r+1}{2}\right|\leqq\frac{1-r}{4}$$

$$s(n)\colon \arg z=\frac{\pi}{n} \qquad \left||z|-\frac{r+1}{2}\right|\leqq\frac{1-r}{4}\left(1-\frac{1}{|n|}\right) \quad (n=\pm1, \pm2,\cdots)$$

$$s(n, n')\colon \arg z=\frac{\pi}{n}+\frac{\theta(n)}{3}\cdot\frac{1}{n'}, \quad \left||z|-\frac{r+1}{2}\right|\leqq\frac{L-r}{4}\left(1-\frac{1}{|n|}\right)\left(1-\frac{1}{|n'|}\right)$$
$$(n'=\pm1, \pm2,\cdots),$$

where $\theta(n)$ is the smaller of the arguments of $s(n)$ and $s(n\pm1)$.

Thus we define the systems $s(n, n',\cdots)$ successively. Let E be the closure of union of such systems, S_1 the domain bounded by E, Γ_1 and Γ_2. By Theorem 5, S_1 is minimal.

Dividing S_1 to two doubly connected domains $S_1^{(1)}$ and $S_1^{(2)}$ by the set E and the circle $|z|=\dfrac{r+1}{2}$. Let E_1 be the part of E involved in the ring $\dfrac{r+1}{2}<|z|<1$. We map $S_1^{(1)}$ conformally onto the ring $r_1<|w|<1$ in the w-plane by $w=f(z)$.

Since each slit of E_1 is the boundary element of $S_1^{(1)}$ in the sense of Carathéodory, E_1 corresponds to a disconnected set on circle $|w|=r_1$ by f. By the principle of reflection, $f(z)$ can be analytically continued to $S_1^{(2)}$.

The image S_2 of S_1 by f is the circular ring with the disconnected set on circle $|w|=r$. By Theorem 2, S_2 is not minimal, while the measure of $f(E)$ is zero.

References

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