37. *Note on the Cluster Sets of the Meromorphic Functions*

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Let $w=f(z)$ be uniform and meromorphic in the arbitrary connected domain D, whose boundary is C and z_0 be the non-isolated boundary point on C. We denote the part of D and C in $|z-z_0|<r$ by D_r and C_r respectively. As usual, we define the cluster sets at z_0 as follows:

Definition 1. *The interior cluster set* $S_{z_0}^{(D)}$ *is defined by* $\bigcap_{r>0}\overline{\mathfrak{D}}_r$, *where* $\mathfrak{D}_r$ *is the set of values taken by* $f(z)$ *in* D_r.

The intersection $\bigcap_{r>0}\mathfrak{D}_r$ *is called the range of values* $R_{z_0}^{(D)}$. *The boundary cluster set* $S_{z_0}^{(C)}$ *is the intersection* $\bigcap_{r>0}M_r$, *where* M_r *is the closure of the union* $\bigcup_{z_0\neq z'\in C_r} S_{z'}^{(D)}$.

Definition 2. *We denote by* $O_{z_0}^{(D)}$ *the set of inner points of* $S_{z_0}^{(D)}$.

K. Noshiro has proved the

Theorem [1, p. 84]. *The set* $R_{z_0}^{(D)}$ *is everywhere dense in the open set* $G=S_{z_0}^{(D)}-S_{z_0}^{(C)}$, *provided that* G *is not empty.*

In this note, we shall prove the following extension of K. Noshiro's theorem, i.e. the

Theorem. *If* $O_{z_0}^{(D)}$ *is not empty, then* $R_{z_0}^{(D)}$ *contains the* G_δ *set* $\bigcap_m O_m$, *where* O_m *is the open set everywhere dense with respect to* $O_{z_0}^{(D)}$.

As its corollary we get the

Corollary. *If* $R_{z_0}^{(D)}$ *is empty, then* $S_{z_0}^{(D)}$ *has no inner point.*

The argument of its proof is due to F. Bagemihl [2].

Proof. By a rational disk we mean an open disk whose radius is a rational number and whose center is a complex number with its real and imaginary parts both rational. The set of rational disks contained in $O_{z_0}^{(D)}$ is enumerable. Let its elements be denoted by D_1, $D_2,\cdots,D_n\cdots$. According to the definition of $S_{z_0}^{(D)}$, for every natural number n, there exists a point $d_n^{(m)}\in D_n$ and a point $z_n^{(m)}\in D$ such that $f(z_n^{(m)})=d_n^{(m)}$, $0<|z_n^{(m)}-z_0|<1/2m$. Hence there exists an open disk $K_n^m\subset D_n$ with $d_n^{(m)}\in K_n^{(m)}$, such that every value in K_n^m is assumed by $f(z)$ in $D\cap C_n^m$, where C_n^m is the disk defined by $|z-z_n^{(m)}|<1/2m$.

Put $O_m=\bigcup_n K_n^m$. It is obvious that O_m is open. O_m is everywhere dense in $O_{z_0}^{(D)}$, because if N is any neighbourhood contained in $O_{z_0}^{(D)}$, then there exists k such that $D_k\subset N$, and since $K_k^m\subset D_k\subset N$, $O_m\cap N$ is not empty.

Put $R=\bigcap_m O_m$, and suppose that $w_0\in R$. Then $w_0\in O_m$ $(m=1, 2,\cdots)$ and hence there exist natural numbers $n_1, n_2,\cdots, n_m,\cdots$ such that $w_0\in K^m_{n_m}$ $(m=1, 2,\cdots)$. Consequently, the value w_0 is assumed by $f(z)$ in each $D\cap C^m_{n_m}$ $(m=1, 2,\cdots)$. If $r>0$ is arbitrary, and μ is a natural number such that $1/\mu<r$, then $C^{\mu}_{n_\mu}$ is a subset of the neighbourhood $|z-z_0|<r$. This means that each element of R is assumed by $f(z)$ in any neighbourhood of z_0. Hence $R\subset R^{(D)}_{z_0}$, which proves our theorem.

References

[1] K. Noshiro: On the singularities of analytic functions, Jap. Jour. Math., **17**, 37–96 (1940).

[2] F. Bagemihl: On the set of values assumed by holomorphic functions near essential singularities, Math. Zeit., **67**, 49–50 (1957).