35. On the Capacitability of Analytic Sets

By Masanori KISHI
Mathematical Institute, Nagoya University
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1. The results, obtained by Choquet [2], on the capacitability were extended by Aronszajn-Smith [1] and the author [3] as follows. Every analytic set in the τ-dimensional Euclidean space is capacitable with respect to the capacity of order α, where $0<\alpha<\tau$ [1]. Let, in general, Ω be a locally compact space, every compact subset of which is metrisable and suppose that a positive symmetric kernel function satisfies Frostman's maximum principle. Then every analytic set in Ω, which is contained in a compact set, is capacitable with respect to the capacity defined by admissible measures [3].

This note will communicate some extensions of these results, details of which will be published later.

2. Let Ω be a locally compact separable metric space, and let Φ be a positive symmetric kernel function which satisfies the following two conditions:

1° the continuity principle, that is, the continuity of the restriction of any potential U^{μ} of a positive measure μ to its carrier S_{μ} implies the continuity of U^{μ} in Ω,

2° when Ω is non-compact, there exists, for any compact subset K and for any positive number ε, a compact subset $L\supset K$ such that $\Phi(P,Q)<\varepsilon$ in $K\times(\Omega-L)$.

Since Ω is separable, there exists an exhaustion $\{\Omega^{(m)}\}$ $(m=1,2,\cdots)$ of Ω such that each $\Omega^{(m)}$ is an open set whose closure is compact, $\Omega^{(m)}\subset\Omega^{(m+1)}$ and $\Omega=\bigcup_{m=1}^{\infty}\Omega^{(m)}$. In the following consideration we take an exhaustion $\{\Omega^{(m)}\}$ of Ω and we fix it. We say that a sequence $\{\mu_n\}$ $(n=1,2,\cdots)$ of positive measures converges vaguely to a positive measure μ when it has the following properties:

(1) it converges vaguely to μ in the ordinary sense,

(2) for each m, the sequence $\{\mu_n^{(m)}\}$ of the restrictions of μ_n to $\Omega^{(m)}$ converges vaguely in the ordinary sense to $\mu^{(m)}$ which coincides with μ in $\Omega^{(m)}$

3. Now let μ be a positive measure whose total measure is finite. Every subset of a set, at which U^{μ} is infinite, is called a polar set. We denote by $\mathfrak{P}$ the family of all polar sets. Obviously $E=\bigcup_{n=1}^{\infty}E_n$ is a polar set, if every E_n is a polar set.

For an arbitrary set X we consider the following families $\mathfrak{F}_X$ and

$\mathfrak{G}_X$ of positive measures:

$$\mathfrak{F}_X=\{\mu>0;\ U^\mu \geqq 1 \text{ except } E\in\mathfrak{P} \text{ in } X\}$$
$$\mathfrak{G}_X=\{\mu>0;\ S_\mu\subset X,\ U^\mu\leqq 1 \text{ in } \Omega\},$$

where the statement, $U^\mu\geqq 1$ except $E\in\mathfrak{P}$ in X, means that the set $E=\{P\in X;\ U^\mu(P)<1\}$ belongs to the family $\mathfrak{P}$. We put

$$f(X)=\inf \mu(\Omega) \quad \text{for all } \mu\in\mathfrak{F}_X$$
$$g(X)=\sup \mu(X) \quad \text{for all } \mu\in\mathfrak{G}_X;$$

obviously f and g are increasing set-functions, that is, if $X\subset Y$, then $f(X)\leqq f(Y)$ and $g(X)\leqq g(Y)$. The inner capacity $\mathfrak{F}\text{-cap}_i(X)$ and the outer capacity $\mathfrak{F}\text{-cap}_e(X)$ of a set X are defined as follows:

$$\mathfrak{F}\text{-cap}_i(X)=\sup f(K) \quad \text{for all compact sets } K\subset X$$
$$\mathfrak{F}\text{-cap}_e(X)=\inf \mathfrak{F}\text{-cap}_i(G) \quad \text{for all open sets } G\supset X.$$

In these definitions we replace $f(K)$ by $g(K)$ and we define $\mathfrak{G}\text{-cap}_i(X)$ and $\mathfrak{G}\text{-cap}_e(X)$ of X. We say that a set X is $\mathfrak{F}[\mathfrak{G}]$-capacitable when $\mathfrak{F}\text{-cap}_i(X)=\mathfrak{F}\text{-cap}_e(X)$ $[\mathfrak{G}\text{-cap}_i(X)=\mathfrak{G}\text{-cap}_e(X)]$ and we denote by $\mathfrak{F}[\mathfrak{G}]\text{-cap}(X)$ the common value of these two capacities. Every open set is $\mathfrak{F}$- and $\mathfrak{G}$-capacitable. Our purpose is to show that every analytic set is $\mathfrak{F}$-capacitable, and hence we may assume that Ω is of positive $\mathfrak{F}$-capacity.

4. Relations between capacities defined above are shown in the following theorems.

Theorem 1. *For any set X we have*

$$\mathfrak{G}\text{-cap}_i(X)\leqq\mathfrak{F}\text{-cap}_i(X) \quad \textit{and} \quad \mathfrak{G}\text{-cap}_e(X)\leqq\mathfrak{F}\text{-cap}_e(X).$$

Theorem 2. *For any set X, $\mathfrak{F}\text{-cap}_i(X)=0$ is equivalent to $\mathfrak{G}\text{-cap}_i(X)=0$.*

This theorem is derived easily from the well-known Evans-Selberg's theorem.

5. Useful theorems in our consideration are the following.

Theorem 3. *A set X is of $\mathfrak{F}$-outer capacity zero if and only if it is a polar set.*

Theorem 4. *For any set X the equality $\mathfrak{F}\text{-cap}_e(X)=f(X)$ holds.*

Corollary. *Every compact set is $\mathfrak{F}$-capacitable.*

This corollary is shown without two conditions 1° and 2°, stated in 2.

Theorem 5. *Suppose that a sequence $\{\mu_n\}$ $(n=1,2,\cdots)$ of positive measures converges vaguely to a positive measure μ and that the total measures $\mu_n(\Omega)$ are bounded. Then it holds that $U^\mu=\lim U^{\mu_n}$ except $E\in\mathfrak{P}$ in Ω.*

From Theorems 4 and 5 follows

Theorem 6. *Suppose that an increasing sequence $\{X_n\}$ of arbitrary sets converges to a set X. Then it holds that $\mathfrak{F}\text{-cap}_e(X)=\lim \mathfrak{F}\text{-cap}_e(X_n)$.*

6. After we have obtained these theorems we apply Choquet's method to prove

Theorem 7. *Every analytic set is* $\mathfrak{F}$*-capacitable.*

Corollary. *If an analytic set is of* $\mathfrak{G}$*-inner capacity zero, then it is of* $\mathfrak{G}$*-outer capacity zero.*

Remark 1. If Φ satisfies Frostman's maximum principle, then $\mathfrak{F}$-capacity coincides with $\mathfrak{G}$-capacity.

Remark 2. $\mathfrak{F}$-$\text{cap}_e(X)$ of a set X coincides with the value defined by $\inf \mu(\Omega)$ for all positive measures such that $U^\mu \geqq 1$ everywhere in X.

References

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[2] G. Choquet: Theory of capacities, Ann. Inst. Fourier, **5**, 131-295 (1955).

[3] M. Kishi: Capacities of borelian sets and the continuity of potentials, Nagoya Math. Jour., **12**, 195-219 (1957).